Some open problems

Related to Spectral Theory of PDOs,

Workshop "Geometry of Eigenvalues and Eigenfunctions",

CRM, Montreal, June 4–8, 2012

Victor Ivrii

Department of Mathematics, University of Toronto

Table of Contents

Introduction

The theory of the asymptotic distribution of eigenvalues of partial dierential operators is one of the central interesting fields of Analysis and is connected to the general spectral theory, the general theory of partial differential operators, geometry (in particular, spectral geometry), dynamics (in particular, the dynamics of the Hamiltonian billiards), number theory.

Introduction

The theory of the asymptotic distribution of eigenvalues of partial dierential operators is one of the central interesting fields of Analysis and is connected to the general spectral theory, the general theory of partial differential operators, geometry (in particular, spectral geometry), dynamics (in particular, the dynamics of the Hamiltonian billiards), number theory.

It also has multiple applications to physics–quantum theory, radiophysics, geophysics, acoustics, theory of oscillations. All these domains will be impacted either by new results, or methods, or problems.

Introduction

The theory of the asymptotic distribution of eigenvalues of partial dierential operators is one of the central interesting fields of Analysis and is connected to the general spectral theory, the general theory of partial differential operators, geometry (in particular, spectral geometry), dynamics (in particular, the dynamics of the Hamiltonian billiards), number theory.
It also has multiple applications to physics–quantum theory, radiophysics, geophysics, acoustics, theory of oscillations. All these domains will be impacted either by new results, or methods, or problems.
Thus, it is one of the most connected and exciting domains of mathematics.

Introduction

The theory of the asymptotic distribution of eigenvalues of partial dierential operators is one of the central interesting fields of Analysis and is connected to the general spectral theory, the general theory of partial differential operators, geometry (in particular, spectral geometry), dynamics (in particular, the dynamics of the Hamiltonian billiards), number theory.

It also has multiple applications to physics–quantum theory, radiophysics, geophysics, acoustics, theory of oscillations. All these domains will be impacted either by new results, or methods, or problems.

Thus, it is one of the most connected and exciting domains of mathematics. IMHO, it is the most connected and exciting domain of mathematics.

IMHO=In My Holy Opinion.

Ivrii' conjecture

In 1980 Victor Ivrii (yes, it is me), fresh after deriving 2-term asymptotics (with the remainder estimate $o(\lambda^{(d-1)/2})$) for an eigenvalue counting function for Laplacians on the manifold X with the boundary with the Dirichlet or Neumann boundary conditions under assumption

Geometric assumption

The set of all periodic geodesic billiards in S^*X has measure 0

Ivrii' conjecture

In 1980 Victor Ivrii (yes, it is me), fresh after deriving 2-term asymptotics (with the remainder estimate $o(\lambda^{(d-1)/2})$) for an eigenvalue counting function for Laplacians on the manifold X with the boundary with the Dirichlet or Neumann boundary conditions under assumption

Geometric assumption

The set of all periodic geodesic billiards in S^*X has measure 0

conjectured that

Conjecture

This assumption always holds for Euclidean domains with $\mathscr{C}^\infty$ boundaries.

Ivrii' conjecture

In 1980 Victor Ivrii (yes, it is me), fresh after deriving 2-term asymptotics (with the remainder estimate $o(\lambda^{(d-1)/2})$) for an eigenvalue counting function for Laplacians on the manifold X with the boundary with the Dirichlet or Neumann boundary conditions under assumption

Geometric assumption

The set of all periodic geodesic billiards in S^*X has measure 0

conjectured that

Conjecture

This assumption always holds for Euclidean domains with $\mathscr{C}^\infty$ boundaries.

Considered by billiardists as an easy game originally it evolved into one of the toughest problems for general as opposed to generic domains and still stands.

IC sidekicks

IC pro

IC holds for Euclidean domains with corners, edges, vertices and conical points etc.

IC sidekicks

IC pro

IC holds for Euclidean domains with corners, edges, vertices and conical points etc. Allow the rest of the boundary be a bit more regular than $\mathscr{C}^1$.

Such singularities now are not an obstacle to derive 2-term asymptotics–under mentioned geometric assumption.

IC sidekicks

IC pro

IC holds for Euclidean domains with corners, edges, vertices and conical points etc. Allow the rest of the boundary be a bit more regular than $\mathscr{C}^1$.

Such singularities now are not an obstacle to derive 2-term asymptotics–under mentioned geometric assumption.

Dima Vassiliev proved that IC holds for any strongly convex domain with analytic boundary.

IC light

IC holds for $X = X_0 \setminus \bigcup_{1\le j\le J} X_j$ where X_j $(j = 0, \dots, J)$ are Euclidean strictly convex domains with non-intersecting analytic boundaries.

We call $x \in X$ absolute looping point if
$\mu_\xi(\{\xi : \exists t \neq 0 : \pi_x(\Phi^t(x,\xi)) = x\}) > 0$.

IC for loops

For Euclidean domain $\mu_x(\text{set of absolute looping points}) = 0$.

Branching trajectories–I

Consider elliptic matrix operator H (no boundary now). Let $\sigma_j(x,\xi)$ $j=1,\dots,J$ be the eigenvalues of its principal symbol (x,ξ). If σ_j have constant multiplicities then propagation of singularities goes independently for each σ_j and geometric condition should be satisfied for each σ_j.

Branching trajectories–I

Consider elliptic matrix operator H (no boundary now). Let $\sigma_j(x,\xi)$ $j=1,\dots,J$ be the eigenvalues of its principal symbol (x,ξ). If σ_j have constant multiplicities then propagation of singularities goes independently for each σ_j and geometric condition should be satisfied for each σ_j. However if σ_j have variable multiplicities then instead of standard equation to bicharacteristics we have

$$\frac{d(x,\xi)}{dt} \in \Gamma(x,\xi)$$

where $\Gamma(x,\xi) \subset T^*_{(x,\xi)}X$ is a convex cone (1-dimensional provided $\sigma_1 \ne \sigma_2$ at (x,ξ)); we sssume for simplicity $J=2$. There is branching of trajectories. However if σ_1, σ_2 are smooth and $\mathcal{H}^n_{\sigma_j}\sigma_k \ne 0$ ($\mathcal{H}_\sigma$ is a Hamiltonian v.f.) there branch takes much less energy and otherwise majority of propagation misses such points. So

Another IC

Variable multiplicity really does not matter for a final result.

Branching trajectories–II

If we consider Laplacian with metrics slightly better than $\mathscr{C}^1$ we can derive $O(\lambda^{(d-1)/2})$ remainder estimate.

Branching trajectories–II

If we consider Laplacian with metrics slightly better than $\mathscr{C}^1$ we can derive $O(\lambda^{(d-1)/2})$ remainder estimate. However Hamiltonian dynamic system has not Lipschitz right-hand expression and from the general theory of ODEs we cannot guarantee unicity of the Hamiltonian (geodesic) flow.

Branching trajectories–II

If we consider Laplacian with metrics slightly better than $\mathscr{C}^1$ we can derive $O(\lambda^{(d-1)/2})$ remainder estimate. However Hamiltonian dynamic system has not Lipschitz right-hand expression and from the general theory of ODEs we cannot guarantee unicity of the Hamiltonian (geodesic) flow.

Or can we taking into account special type of equation?

Monotonicity

Eigenvalues (and therefore eigenvalue counting functions) depend monotonically on operator.

Monotonicity

Eigenvalues (and therefore eigenvalue counting functions) depend monotonically on operator. For Dirichlet Laplacians they depend monotonically on the domain.

Monotonicity

Eigenvalues (and therefore eigenvalue counting functions) depend monotonically on operator. For Dirichlet Laplacians they depend monotonically on the domain.

What about $e(x,x,\lambda)$ where $e(x,y,\lambda)$ is a Schwartz kernel of spectral projector $E(\lambda)$?

Monotonicity

Eigenvalues (and therefore eigenvalue counting functions) depend monotonically on operator. For Dirichlet Laplacians they depend monotonically on the domain.

What about $e(x,x,\lambda)$ where $e(x,y,\lambda)$ is a Schwartz kernel of spectral projector $E(\lambda)$? Or $\operatorname{Tr} BE(\lambda)$ where B is a positive operator?

Monotonicity

Eigenvalues (and therefore eigenvalue counting functions) depend monotonically on operator. For Dirichlet Laplacians they depend monotonically on the domain.

What about $e(x,x,\lambda)$ where $e(x,y,\lambda)$ is a Schwartz kernel of spectral projector $E(\lambda)$? Or $\operatorname{Tr} BE(\lambda)$ where B is a positive operator?

Is there anything replacing monotonicity?

Spacing

Statistics of eigenvalues λ_n $(\ldots \leq \lambda_n \leq \lambda_{n+1} \leq \ldots)$ of operator H is pretty well known.

Spacing

Statistics of eigenvalues λ_n $(\ldots \le \lambda_n \le \lambda_{n+1} \le \ldots)$ of operator H is pretty well known.

Statistics of consecutive spacings $(\lambda_{n+1} - \lambda_n)$ for $d \ge 2$ is practically unknown and physicists who are very interested in it study instead matrix models.

Spacing

Statistics of eigenvalues λ_n $(\ldots \le \lambda_n \le \lambda_{n+1} \le \ldots)$ of operator H is pretty well known.

Statistics of consecutive spacings $(\lambda_{n+1} - \lambda_n)$ for $d \ge 2$ is practically unknown and physicists who are very interested in it study instead matrix models.

Statistics of consecutive spacings

Develop methods to deal with distribution of spacings.

Spacing

Statistics of eigenvalues λ_n $(\ldots \le \lambda_n \le \lambda_{n+1} \le \ldots)$ of operator H is pretty well known.

Statistics of consecutive spacings $(\lambda_{n+1} - \lambda_n)$ for $d \ge 2$ is practically unknown and physicists who are very interested in it study instead matrix models.

Statistics of consecutive spacings

Develop methods to deal with distribution of spacings.

Statistics of all spacings $(\lambda_m - \lambda_n)$ follows from statistics of λ_n.

More problems

Many open problems in my forthcoming Monster Size Monograph:
http://weyl.math.toronto.edu:8888/victor_ivrii/research/future-book/

And further, by these, my son, be admonished: of making many books there is no end; and much study is a weariness of the flesh.

Ecclesiastes 12, King James Version (KJV)