2D- and 3D-Magnetic Schrödinger Operator: Short Loops and Pointwise Spectral Asymptotics

Workshop "Spectral Gap in Dynamical Systems, Number Theory and PDEs", Peyresq, France, May 30–June 3, 2011

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June 1, 2011 – the Day when Internet died

Table of Contents

Framework

Consider Magnetic Schrödinger Operator

$$H := h^2 D_1^2 + (hD_2 - \mu x_1)^2 + h^2 \mathrm{D}_3^2 + V(x) \tag{1}$$

but the similar results hold for a more general operator

$$H := \sum_{j,k} \bigl(hD_j - \mu A_j(x)\bigr) g^{jk} \bigl(hD_k - \mu A_k(x)\bigr) + V(x) \tag{2}$$

provided

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Magnetic intensity $\mathbf{F} = \nabla \times \mathbf{A}$ is disjoint from 0.

Here $h \ll 1$ and $\mu \gg 1$ are semiclassical parameter and magnetic field intensity. (1) is a canonical form of such operator with Euclidean metrics and constant magnetic field.

We assume that this operator is self-adjoint. Let $E(\tau)$ be its spectral projector, $e(x,y,\tau)$ its Schwartz kernel and we are interested in pointwise spectral asymptotics

$$e(x,x,\tau) \qquad \text{as} \quad h\to +0, \mu\to +\infty. \tag{3}$$

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$$\int e(x,x,\tau)\psi(x)\,dx \qquad \text{as} \quad h\to +0, \mu\to +\infty \tag{4}$$

with $\psi\in C_0^\infty$ because from it one can assemble an eigenvalue counting function given by the same expression with $\psi=1$.

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Remark

We take $|\tau|\le c$ and often $\tau=0$.

Simple scaling $x \to \mu x$, $h \to \mu h$, $\mu \to 1$ implies

$$e(x,x,\tau) = \mathcal{N}^{\mathsf{W}}(x,\tau) + O(\mu h^{1-d}) \qquad \text{as} \quad \mu h \le 1. \tag{5}$$

where $\mathcal{N}^{\mathsf{W}} = \operatorname{const}(\tau - V)_+^{d/2}$ is a Weyl expression.

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$$e(x,x,\tau) = \mathcal{N}^{\mathsf{MW}}(x,\tau) := (2\pi)^{-1}\mu h^{-1}\sum_j \theta(\tau - V(x) - (2j+1)\mu h) \tag{6}$$

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with Heaviside function θ. As τ goes through Landau level it jumps by μh^{-1}. However under certain non-degeneracy assumptions remainder estimate could be much better and this is the subject of the talk.

We discuss 2D-case now

From the dynamical point of view a pilot-model operator with no electric field is bad because all classical trajectories are periodic with periods $\asymp \mu^{-1}$ (they are circles with radii $\asymp \mu^{-1}$ (as $\tau - V \asymp 1$)).

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From the dynamical point of view a pilot-model operator with no electric field is bad because all classical trajectories are periodic with periods $\asymp \mu^{-1}$ (they are circles with radii $\asymp \mu^{-1}$ (as $\tau - V \asymp 1$)). However let add constant electric field. Then Hamiltonian trajectories (their x-projections) are prolate cycloids

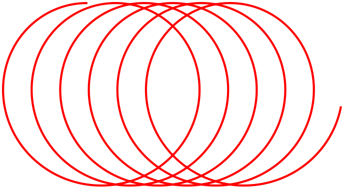

Figure: Drift is orthogonal to electric field and its speed is $\asymp \mu^{-1}\alpha$ where α is the electric intensity

Electric field breaks periodicity (we need to check that it is the case in the "quantum" sense) but assuming that our domain contains $B(0,2)$ where operator is "good" and ψ is supported in $B(0,1)$ we know that there are no periods $\le \epsilon\mu$ so actually our remainder estimate in local spectral asymptotics $O(T^{-1}h^{-1})$ improves from $O(\mu h^{-1})$ to $O(\mu^{-1}h^{-1})$ as T improves from $\epsilon\mu^{-1}$ to $\epsilon\mu$.

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Theorem (Old theorem)

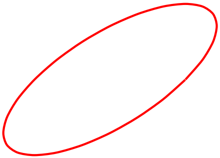
As ∇V disjoint from 0

$$\int e(x,x,0)\psi(x)\,dx = \int \mathcal{N}^{\mathsf{MW}}(x,0)\psi(x)\,dx + O(\mu^{-1}h^{-1}) \qquad \mu h \le 1. \tag{7}$$

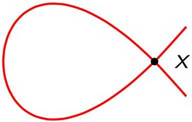
Further for $\mu h \ge 1$ remainder estimate is $O(1)$ as we consider Schrödinger-Pauli operator (subtract $(2n+1)\mu h$ from H) and the principal part is $\asymp \mu h^{-1}$.

But what about pointwise asymptotics?

There are new villains - loops when trajectory returns to the same point x but from the different direction:

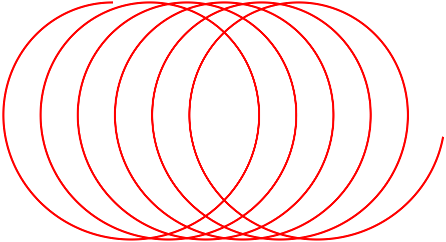


(a) periodic

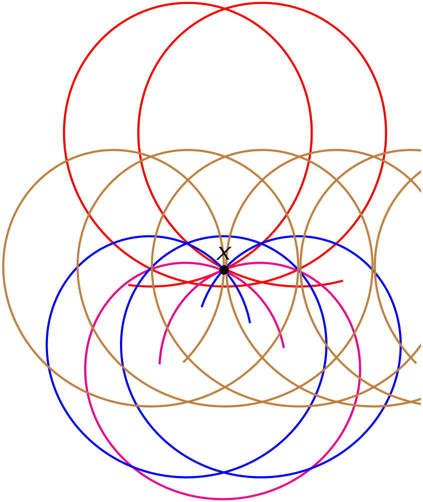


(b) loop

and there are plenty of loops in our case – and many of them are short!



But we don't care about many self-intersections on a single trajectory: after ± 1, $\pm 2, \ldots$ rotations



However there are plenty of trajectories looping in the given point x.

2D case

We consider a pilot-model with a constant electric field (i.e. linear V)

$$H = \bar{H} := h^2 D_1^2 + (hD_2 - \mu x_1)^2 + 2\alpha x_1 \tag{8}$$

with $\alpha \asymp 1$. Results for general operators are similar.

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We need to consider $U(x,x,t)$ where $U(x,y,t)$ is the Schwartz kernel for propagator $e^{ih^{-1}tH}$. If we rescale $x\mapsto \mu x$, $t\mapsto \mu t$ (and $T=\epsilon\mu^2$), $h\mapsto \hbar=\mu h$ we can write precisely

$$U(x,y,t)=(2\pi h)^{-1}\mu\int \mathrm{u}(x_1,y_1;\eta,t)e^{ih^{-1}(x_2-y_2)\eta}\,d\eta \tag{9}$$

with $\mathrm{u}(x_1,y_1;\eta,t)$ the Schwartz kernel of $e^{i\hbar^{-1}t\mathbf{a}}$ with 1D-harmonic oscillator

$$\mathbf{a}=\hbar^2D_1^2+(x_1-\eta)^2+2\alpha\mu^{-1}x_1=\\ \underbrace{\hbar^2D_1^2+(x_1-\eta+\alpha\mu^{-1})^2}_{\bar{\mathbf{a}}}+\mu^{-1}\alpha\underbrace{(2\eta-\alpha\mu^{-1})}_{\zeta(\eta)}. \tag{10}$$

For the harmonic oscillator $\mathbf{b} = D^2 + x^2$ the Schwartz kernel of $e^{it\mathbf{b}}$ is known exactly and after calculations we arrive to

$$U(x,y,t) = i(4\pi)^{-1}\mu h^{-1}\csc(t)\, e^{i\hbar^{-1}\phi(x,y,t)} \tag{11}$$

with

$$\phi := -\frac{1}{4}\cot(t)(x_1-y_1)^2 + \frac{1}{2}(x_1+y_1+2\alpha\mu^{-1})(x_2-y_2+2t\alpha\mu^{-1}) - \frac{1}{4}\cot(t)(x_2-y_2+2t\alpha\mu^{-1})^2 - t\alpha^2\mu^{-2} \tag{12}$$

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and then

$$U(0,0,t) \equiv i(4\pi)^{-1}\mu h^{-1}\csc(t)\, e^{i\hbar^{-1}\bar{\phi}(t)} \tag{13}$$

with

$$\bar{\phi}(t) := t^2\alpha^2\mu^{-2}\cot(t) + \alpha^2\mu^{-2}t. \tag{14}$$

It allows us to write an exact formula for

$$F_{t\to\mu^{-1}h^{-1}t}\bar{\chi}_T(t)U(0,0,t)\,dt = \\ 2i(4\pi)^{-2}\mu h^{-1}\int \csc(t)\, e^{i\hbar^{-1}(\bar{\phi}(t)-\mu^{-2}t\tau)}\bar{\chi}_T(t)\,dt \qquad (15)$$

($\bar{\chi}\in C_0^\infty([-1,1])$, $\bar{\chi}_T(t)=\bar{\chi}(t/T)$ and we can try to apply a stationary phase to it; then we get

$$t_k=-t_{-k},\quad t_k\sim \pi k,\quad \sin(t_k)\sim \alpha\mu^{-1}\tau^{-\frac{1}{2}}\pi k \qquad (16)$$

and

Remark

The number of stationary points is $\sim 2\pi^{-1}|\alpha|^{-1}\mu\tau^{-\frac{1}{2}}$.

Remark

Stationary phase method works for all k only if $\mu \le h^{-\frac{1}{2}}$. As $h^{-\frac{1}{2}} \le \mu \le h^{-1}$ this method works only for $k: |k| \ge \mu^2 h$.

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However one can prove

Theorem

For $d = 2$

$$|F_{t\to\mu^{-1}h^{-1}t}\bar{\chi}_T(t)U(0,0,t)| \le C\mu h^{-1} + C\mu^{\frac{5}{2}}h^{-\frac{1}{2}}. \tag{17}$$

Here the main contribution into second term is delivered by points with $k \asymp \mu$. This estimate holds also for Schrödinger-Pauli operator as $\mu \ge h^{-1}$. This theorem holds also for general operator (1) as $|\nabla V| \asymp 1$.

Due to the standard Tauberian theory (with $T = \epsilon\mu^2$) the above theorem instantly implies:

Theorem

For $d = 2$

$$|e(0,0,\tau) - \underbrace{\mu^{-1}h^{-1}\int_{-\infty}^{\tau}\Big(F_{t\to\mu^{-1}h^{-1}t}\bar{\chi}(t)U(0,0,t)\Big)\,d\tau}_{\textit{Tauberian expression}}| \le$$

$$C\mu^{-1}h^{-1} + C\mu^{\frac{1}{2}}h^{-\frac{1}{2}}. \tag{18}$$

Note that the second term dominates as $\mu \ge h^{-\frac{1}{3}}$.

Now we need to calculate Tauberian expression.

Theorem

As $1 \le \mu \le h^{-\frac{1}{2}}$

$$|e(x,x,0) - \mathcal{N}_x^{\mathsf{W}}(0)| \le C\mu^{-1}h^{-1} + C\mu^{\frac{1}{2}}h^{-\frac{1}{2}} + C\mu^2 h^{-\frac{1}{2}} \tag{19}$$

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Theorem

As $1 \le \mu \le h^{-\frac{1}{2}}$

$$|e(x,x,0) - \bigl(\mathcal{N}_x^{\mathsf{W}}(0) + \mathcal{N}_{x,\mathrm{corr}(r)}^{\mathsf{W}}(0)\bigr)| \le \\ C\mu^{-1}h^{-1} + C\mu^{\frac{1}{2}}h^{-\frac{1}{2}} + C\mu h^{-1}(\mu^2 h)^{r+\frac{1}{2}} + \\ C\begin{cases} \Bigl(h^{-1}\bigl(h\mu^{\frac{5}{2}}\bigr)^{r+\frac{1}{2}} + \mu^{\frac{1}{3}}h^{-\frac{2}{3}}\Bigr) & \textit{as} \quad \mu \le h^{-\frac{2}{5}}, \\ \mu^{\frac{5}{3}}h^{-\frac{1}{3}} & \textit{as} \quad \mu \ge h^{-\frac{2}{5}}. \end{cases} \tag{20}$$

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Here we take r terms in the stationary phase method but the remainder estimate is good only as $\mu \le h^{\delta-\frac{1}{2}}$ and we need to take $r = r(\delta)$ terms to eliminate the third term in the right-hand expression.

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$$|e(x,x,0)-\bigl(\mathcal{N}_x^{\mathsf{W}}(0)+\mathcal{N}_{x,\mathrm{corr}(r)}^{\mathsf{W}}(0)\bigr)|\le \\ C\mu^{-1}h^{-1}+C\mu^{\frac{1}{2}}h^{-\frac{1}{2}}+C\mu h^{-1}(\mu^2 h)^{r+\frac{1}{2}}+ \\ C\begin{cases} \Bigl(h^{-1}(h\mu^{\frac{5}{2}})^{r+\frac{1}{2}}+\mu^{\frac{1}{3}}h^{-\frac{2}{3}}\Bigr) & \textit{as}\quad \mu\le h^{-\frac{2}{5}},\\ \mu^{\frac{5}{3}}h^{-\frac{1}{3}} & \textit{as}\quad \mu\ge h^{-\frac{2}{5}}. \end{cases} \tag{20}$$

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Theorem

(i) As $\mu \ge h^{-\frac{1}{3}}$

$$|e(x,x,\tau)-\bar{e}_x(x,x,\tau)| \le C\mu^{\frac{1}{2}}h^{-\frac{1}{2}} + C\begin{cases} h^{-1}\mu^{\frac{1}{2}} & \text{as } \ \mu \le h^{-\frac{1}{2}}, \\ \mu^{-\frac{1}{2}}h^{-\frac{3}{2}} & \text{as } \ \mu \ge h^{-\frac{1}{2}} \end{cases} \tag{21}$$

where $\bar{e}_x(x,x,\tau)$ is calculated for a pilot-model approximating general operator at point x..

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For results as $h^{-\frac{1}{3}} \le \mu \le h^{-1}$ see main text [1] for sharper asymptotics but they include some correction.

3D case

Consider now 3D case. It is related to 2D case but is drastically different.

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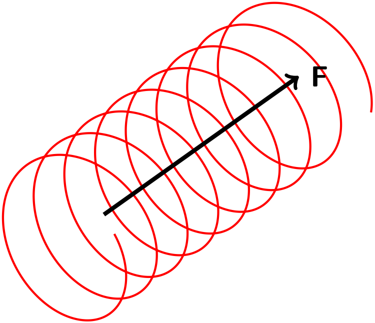
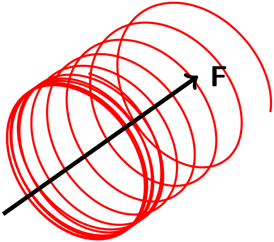


(e) Helix



(f) Perturbed helix

They have double effect.

They have double effect. First, because speed along magnetic lines could be $\asymp 1$, $T \asymp 1$ and $T \asymp \mu$ before/after rescaling. So, our best possible remainder estimate could be $O(h^{-2})$ as $\mu h \le 1$ and $O(\mu h^{-1})$ as $\mu h \ge 1$ for Schrödinger-Pauli operator. Second, movement along magnetic lines usually breaks periodicity:

Theorem (Old theorem)

$$|\int e(x,x,0)\psi(x)\,dx - \int \mathcal{N}^{\mathsf{MW}}(x,0)\psi(x)\,dx| \le Ch^{-2} + C\mu h^{-1-\delta} \qquad \mu h \le 1. \tag{22}$$

where

$$\mathcal{N}^{\mathsf{MW}} := \frac{1}{4\pi}^2 \sum_j \Bigl(\tau - (2j+1)\mu h f - V\Bigr)_+^{\frac{1}{2}} f \mu h^{-2}, \tag{23}$$

and under very weak non-degeneration assumption one can take $\delta = 0$.

Theorem (No non-degeneracy assumption)

$$|e(x,x,0) - \mathcal{N}^{\mathsf{W}}(x,0)| \le Ch^{-2} + C\mu^{\frac{3}{2}}h^{-\frac{3}{2}}. \tag{24}$$

In particular, as $\mu \le h^{-\frac{1}{3}}$ remainder estimate is $O(h^{-2})$.

Consider a pilot-model in $\mathbb{R}^3$

$$H = \bar{H} := h^2 D_1^2 + (hD_2 - \mu x_1)^2 + h^2 D_3^2 + 2\alpha x_1 + 2\beta x_3. \tag{25}$$

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Then

$$U_{(3)}(x, y, t) = U_{(2)}(x', y', t) U_{(1)}(x_3, y_3, t) \tag{26}$$

where $U_{(2)}$ is former U, $x' = (x_1, x_2)$ etc and $U_{(1)}$ is a constructed for 1D-operator $h^2 D_3^2 + 2\beta x_3$:

$$U_{(1)}(x_3, y_3, t) = \frac{1}{2}(2\pi h t)^{-\frac{1}{2}} \exp\Bigl(i h^{-1} \bigl(\beta t (x_3 + y_3) + \frac{1}{8} t^{-1} (x_3 - y_3)^2 - \frac{2}{3} \beta^2 t^3 \bigr) \Bigr); \tag{27}$$

in particular

$$U_{(1)}(x_3, x_3, t) = \frac{1}{2}(2\pi h |t|)^{-\frac{1}{2}} \exp\Bigl(i h^{-1} \bigl(2\beta t x_3 - \frac{2}{3} \beta^2 t^3 \bigr) \Bigr). \tag{28}$$

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Then playing with oscillatory integrals we prove for the pilot-model (but factor $|t|^{-\frac{1}{2}}$ plays a crucial role) and generalize for general operator we get

Theorem

Let $\mu h \le 1$ and

$$|\nabla_{\perp \mathbf{F}} V/F| \asymp 1. \tag{29}$$

Then

$$|F_{t\to\mu^{-1}h^{-1}\tau}\bar{\chi}_T(t)\Gamma_x \mathrm{U}| \le C\mu h^{-2} + C\mu^{\frac{5}{2}} h^{-1}\bigl(1 + |\log \mu h)|\bigr) \tag{30}$$

and the Tauberian remainder estimate is $O(h^{-2} + \mu^{\frac{3}{2}} h^{-1}(1 + |\log \mu h|))$.

Theorem

As $\mu \le h^{-\frac{1}{2}}$ *under non-degeneracy condition* (29)

$$|e(x,x,0) - \mathcal{N}_x^{\mathsf{W}}(x,x,0)| \le Ch^{-2} + C\mu^{\frac{5}{2}}h^{-1} \tag{31}$$

and

$$|e(x,x,0) - \mathcal{N}_x^{\mathsf{W}}(x,x,0) - \mathcal{N}_{x,\mathsf{corr}(r)}^{\mathsf{W}}| \le Ch^{-2} + C\mu^{\frac{5}{2}}h^{-1}(\mu^2 h)^r. \tag{32}$$

What to do as $\mu \ge h^{\delta-\frac{1}{2}}$?

The same as in 2D-case: approximation by a pilot-model with linear V. Again we estimate an error.

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Theorem

Let $\mu h \geq 1$ and (29) be fulfilled. Then for Schrödinger-Pauli operator (we subtract μhf)

$$|F_{t\to\mu^{-1}h^{-1}\tau}\bar{\chi}_T(t)\Gamma_x \mathrm{U}| \leq C\mu^2 h^{-\frac{3}{2}} \tag{33}$$

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Remark

As $d = 2, 3$ and $\mu h \geq 1$ there are simple representations for $U(x,y,t)$ based on Hermite polynomials and simple formulae for $\bar{e}_x(x,x,0)$.

All details again on [1].

Reference

Victor Ivrii, Microlocal Analysis, Sharp Spectral Asymptotics and Applications,
http://www.math.toronto.edu/ivrii/futurebook.pdf
Ch. 13 for local spectral asymptotics and Chapter 16 (sections 16.1–16.2, and 16.5–16.6) for pointwise spectral asymptotics.