Magnetic Schrödinger Operator near Boundary

First Joint Meeting:
American Mathematical Society & Sociedad de Matemtica de Chile,

December 15–18, 2010, Pucón, Chile

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December 17, 2010

Introduction

We consider Magnetic Schrödinger operator

$$\bar{H} = \bar{H}_{h,\mu} := h^2 D_1^2 + (hD_2 - \mu x_1)^2 + V(x) \tag{1}$$

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in half-plane $X = \{x \in \mathbb{R}^2, x_1 > 0\}$. In fact we can consider more general case (non-Euclidean metrics, different vector-potential and non-constant magnetic field, general domains).
Here $h \ll 1$ is Planck constant, $\mu \gg 1$ is a magnetic intensity and we are interested in local spectral asymptotics i.e. asymptotics of

$$\int_X e(x,x,0)\psi(x)\,dx \tag{2}$$

where $e(x,y,\tau) = e_{h,\mu}(x,y,\tau)$ is the Schwartz kernel of the spectral projector of H and $\psi \in C_0^\infty(\mathbb{R}^2)$ is a fixed.

Classical Dynamics

Assume first that $V(x) = -\tau =$ const (no electric field). Then Hamiltonian trajectories (i.e. trajectories of the classical Hamiltonian - but we are talking about rather their x-projections) on energy level 0

$$\bar{H}(x,\xi) = \bar{H}_\mu(x,\xi) = \xi_1^2 + (\xi_2 - \mu x_1)^2 + V(x) \tag{3}$$

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Trajectories which meet the boundary have rather different fate.

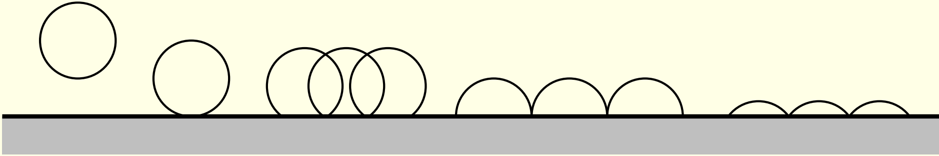
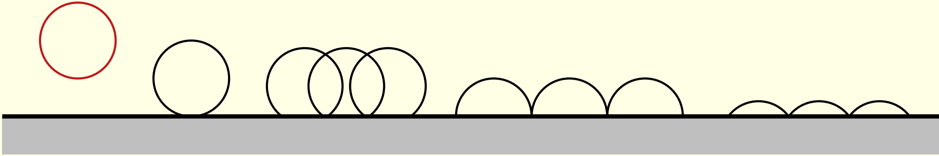


Figure: $V = \text{const}$



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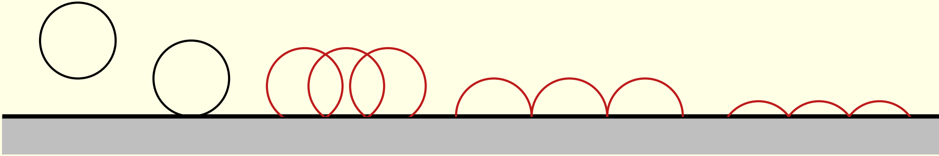

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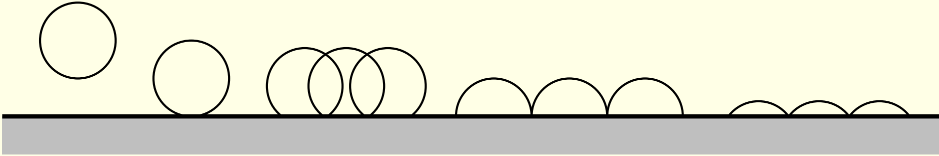

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Physicists talk about bulk electrons and edge electrons.
We talk about inner movement and hop-movement.

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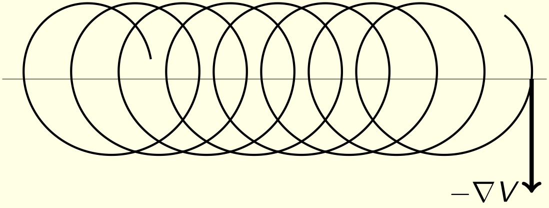

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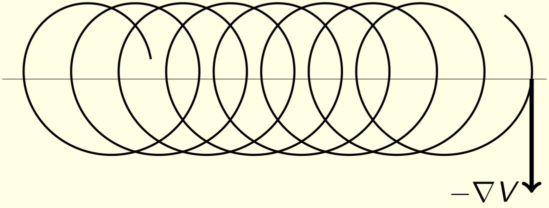

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In general magnetic drift is along level lines of V.

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In the quantum dynamics analysis, the most difficult is an analysis in the transitional zone where the uncertainty principle does not allow to determine if the trajectory reflect from the boundary or barely misses it.

Meanwhile V_{x_1} has more subtle effect. As hop-speed is larger than $C_0\mu^{-1}$ magnetic drift with respect to x_2 has no qualitative effect. However there are no hops inside. Therefore as $V_{x_1} \asymp 1$ we have two rather different cases:

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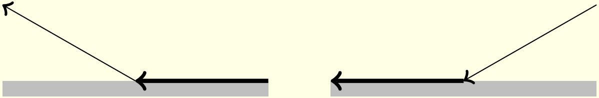
In particular as $V_{x_2} < 0$ hop-trajectories are torn from the boundary and begin the drift movement while as $V_{x_2} > 0$ drift-trajectories collide with the boundary and begin hop-movement.



(a) $V_{x_1} < 0, V_{x_2} < 0$ (b) $V_{x_1} < 0, V_{x_2} > 0$

Figure: Bold lines show hop movement and thin lines show drift movement which is along level lines of V.

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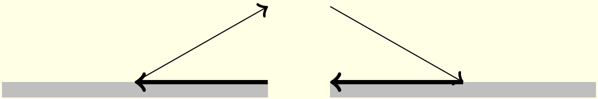

(a) $V_{x_1} > 0, V_{x_2} < 0$ (b) $V_{x_1} > 0, V_{x_2} > 0$

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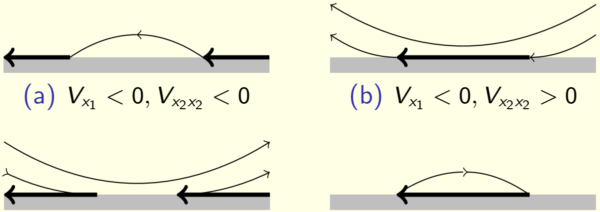
Assume now that V_{x_2} vanishes at some point $\bar{x}$ but $V_{x_2x_2} \neq 0$. Then repeating analysis of the previous example we arrive to the following four pictures:



(a) $V_{x_1} < 0, V_{x_2x_2} < 0$ (b) $V_{x_1} < 0, V_{x_2x_2} > 0$

(c) $V_{x_1} > 0, V_{x_2x_2} < 0$ (d) $V_{x_1} > 0, V_{x_2x_2} > 0$

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Again this is consistent with the fact that drift trajectories are level curves of V: in cases (a), (d), (b)–(c) point $\bar{x}$ is a local maximum, minimum and minimax respectively.

Spectral Asymptotics: $\mu h \lesssim 1$

If $X = \mathbb{R}^2$ and $V(x) = \text{const}$ then operator has the purely point spectrum of infinite multiplicity located at Landau levels $(2n+1)\mu h + V$ and

$$e(x,x,\tau) = \mathsf{N}^{\mathsf{MW}} := (2\pi)^{-1}\left\lfloor \frac{(\tau - V - 1)}{2\mu h}\right\rfloor \times \mu h^{-1}. \tag{4}$$

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In the general case remainder estimate $O(\mu h^{-1})$ could be easily deducted from the classical case $\mu \lesssim 1$ by rescaling method.

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Assume that ψ is supported in the vicinity of the boundary (∂X) and that

$$\nabla V \neq 0 \qquad \text{on} \quad \operatorname{supp}\psi; \tag{5}$$

surely it is fulfilled in the generic case.

Then starting from the results when $\operatorname{supp}\psi \cap \partial X = \emptyset$ by rescaling method one can prove that the contribution of the zone $\{x : \operatorname{dist}(x, \partial X) \ge C_0\mu^{-1}\}$ to the remainder is $O(h^{-1})$.

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Meanwhile contribution of zone $\{x : \operatorname{dist}(x, \partial X) \leq C_0\mu^{-1}\}$ is also $O(\mu^{-1} \times \mu h^{-1})$ where factor μ^{-1} is the volume of this zone and $\mu = T^{-1}$.

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We assume also that $|\tau| \leq C_0$ and

$$V(x) - \tau < 0 \qquad \text{on} \ \ \operatorname{supp}\psi; \tag{6}$$

one can get rid off it later.

Theorem

As $\mu h \lesssim 1$ *under conditions* (5)–(6)

$$\mathsf{R}(\tau) := \int_X \Bigl(e_*(x,x,\tau) - N^{\mathsf{MW}}(x,\tau)\Bigr)\psi(x)\,dx \pm (2\pi h)^{-1}\int_{\partial X} V^{\frac{1}{2}}(x)\psi(x)dx_2 = O\bigl(\mu^{-1}h^{-1} + (\mu h)^{\frac{3}{2}}h^{-1-\delta}\bigr) \tag{7}$$

with arbitrarily small exponent $\delta > 0$ *where* $\pm$ *when* $* = \mathsf{D}, \mathsf{N}$ *(Dirichlet/Neumann boundary condition).*

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The proof is based on the fact that in inner zone condition (5) breaks periodicity and in the boundary zone hop-movement breaks periodicity; however the width of the transitional zone is $\asymp \bigl(\mu^{-2} + (\mu h)^{\frac{3}{2}}\mu^{-1}h^{-\delta}\bigr)$ (analysis based on the microlocal uncertainty principle).

To improve (7) we need to break periodicity in the transitional zone. This happens under assumption

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As $\mu h \lesssim 1$ under conditions (6), (8)

$$\mathsf{R} := \int_X \Bigl(e_*(x,x,0) - N^{\mathsf{MW}}(x,0) \Bigr) \psi(x)\, dx - \int_{\partial X} \mathsf{N}^{\mathsf{MW}}_{*,b} \psi(x) dx_2 = O\bigl(\mu^{-1} h^{-1}\bigr) \tag{9}$$

where exact formula for $\mathsf{N}^{\mathsf{MW}}_{*,b}$ could be found in [1].

It involves eigenvalues of an auxiliary operator (15).

In [1] also deduct asymptotics with the remainder estimates better than (7) but worse than (9) when conditions (5), (6) and

$$\nabla_{\partial X} V = 0 \implies \nabla^2_{\partial X} V \neq 0 \tag{10}$$

with remainder estimates depending on signs on $\partial_{x_1} V$ and $\partial^2_{x_2} V$.

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The main part of these asymptotics involve eigenvalues of the auxiliary 1-dimensional operator (15) (see below) depending on parameter.

Spectral Asymptotics: Superstrong Magnetic Field

As $\mu h \gtrsim 1$ the spectrum shifts up and instead of $\tau = 0$ we assume that

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On the other hand, if τ does not belong to the spectral gap (on $\operatorname{supp}\psi$) then under condition (5)

$$\mathsf{R} := \int_X \Bigl(e(x,x,0) - N^{\mathsf{MW}}(x,0)\Bigr)\psi(x)\,dx = O(1). \tag{14}$$

Situation becomes more complicated near the boundary. Consider auxiliary 1-dimensional operator

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Let $\lambda_{\mathsf{D},n}(\eta)$ and $\lambda_{\mathsf{N},n}(\eta)$ $(n = 0, 1, 2, \ldots)$ be eigenvalues $L_{\mathsf{D}}(\eta)$ and $L_{\mathsf{N}}(\eta)$ respectively.

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In particular, $\lambda_{\mathsf{D},n}(\eta)\searrow(2n+1)$ and $\lambda_{\mathsf{N},n}(\eta)\nearrow(2n+1)$ as $\eta\to+\infty$. Further

$$|\partial_\eta^j\lambda_{*,n}(\eta)|\asymp\eta^{2n+1+j}e^{-\eta^2}\qquad\text{as}\ \ \eta\to+\infty. \tag{17}$$

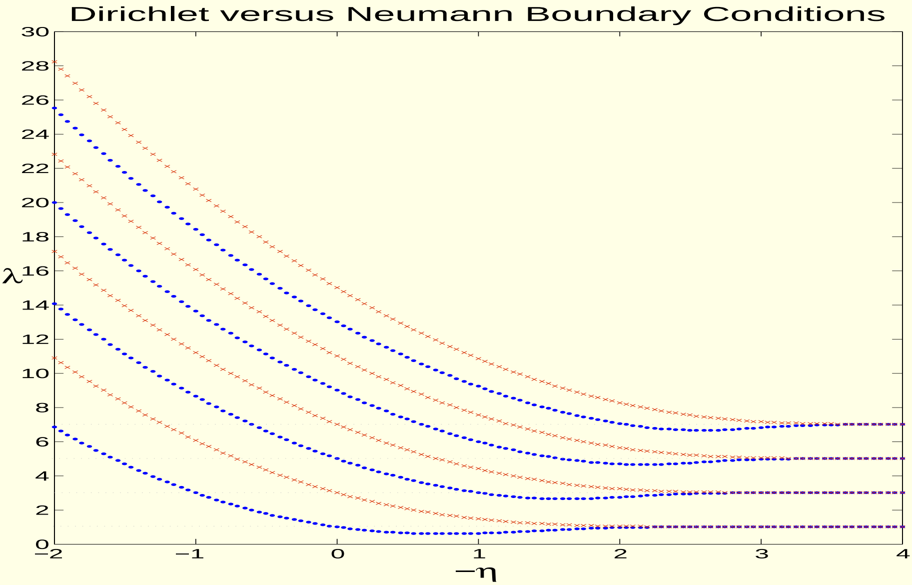

Figure: Plots of $\lambda_{\mathrm{N},n}(\eta)$ and $\lambda_{\mathrm{D},n}(\eta)$. Calculated by Matlab: Courtesy of Dr. M. Chugunova.

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As τ belongs to the (former) *n*-th spectral gap on suppψ, this formula is

$$(2\pi)^{-1}\mu h^{-1}(n-1)+(2\pi)^{-1}\mu^{\frac{1}{2}}h^{-\frac{1}{2}}\int_{\partial X}\int_{\eta:\mu h\lambda_{*,n}+V(x)<0}\psi(x)\,dx_2 d\eta. \quad (20)$$

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Lower bound is a bit fuzzy due to uncertainty principle.

V. Ivrii, Schrödinger Operator with Strong Magnetic Field near Boundary, http://arxiv.org/abs/1005.0244
extended version in Chapter 15 of Future Book
http://www.math.toronto.edu/ivrii/futurebook.pdf.