Magnetic Schrödinger Operator: Geometry, Classical and Quantum Dynamics and Spectral Asymptotics

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I will consider Magnetic Schrödinger operator

\[ H = \frac{1}{2} \left( \sum_{j,k} P_j g^{jk}(x) P_k - V \right), \quad P_j = hD_j - \mu A_j \]  

where \( g^{jk}, A_j, V \) are smooth real-valued functions of \( x \in \mathbb{R}^d \) and \( (g^{jk}) \) is positive-definite matrix, \( 0 < h \ll 1 \) is a Planck parameter and \( \mu \gg 1 \) is a coupling parameter. I assume that \( H \) is self-adjoint operator.
I will consider Magnetic Schrödinger operator

\[ H = \frac{1}{2} \left( \sum_{j,k} P_j g^{jk}(x) P_k - V \right), \quad P_j = hD_j - \mu A_j \tag{1} \]

where \( g^{jk}, A_j, V \) are smooth real-valued functions of \( x \in \mathbb{R}^d \) and \( (g^{jk}) \) is positive-definite matrix, \( 0 < h \ll 1 \) is a Planck parameter and \( \mu \gg 1 \) is a coupling parameter. I assume that \( H \) is self-adjoint operator.

2-dimensional magnetic Schrödinger is very different from 3-dimensional, all others could be close to one of these cases but are more complicated.
I am interested in the geometry of magnetic field,
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I am interested in the geometry of magnetic field, classical and quantum dynamics associated with operator (1) and spectral asymptotics

$$\int e(x, x, 0)\psi(x) \, dx$$

as $h \to +0$, $\mu \to +\infty$ where $e(x, y, \tau)$ is the Schwartz kernel of the spectral projector of $H$ and $\psi(x)$ is cut-off function. Everything is assumed to be $C^\infty$. 

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Magnetic Schrödinger Operator
Magnetic field is described by a form

\[ \sigma = d \left( \sum_k A_k dx_k \right) = \frac{1}{2} \sum_{j,k} F_{jk} dx_j \wedge dx_k \] (3)

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So \( \sigma \) does not change after gauge transformation \( \vec{A} \mapsto \vec{A} + \vec{\nabla} \phi \) and this does not affect other objects I am interesting in as well. I am discussing local things and Aharonov-Bohm effect which demonstrates that knowledge of \( \sigma, g_{jk}, V \) is not sufficient to characterize spectral properties of \( H \) is beyond my analysis.
If $\sigma$ is of maximum rank $2r = 2\lfloor d/2 \rfloor$ one can reduce it locally to the Darboux canonical form

$$\sigma = \sum_{1 \leq j \leq r} dx_{2j-1} \wedge dx_{2j}.$$ (5)

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As $d = 2$ generic form $\sigma$ has a local canonical form

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\( \Lambda = \{ x \in \Sigma, \text{Ker} \, F(x) \subset T_x \Sigma \} \) is submanifold of dimension 1. As \( \bar{x} \in \Sigma \setminus \Lambda \) \( \dim(\text{Ker} \, F(x) \cap T_{\bar{x}} \Sigma) = 1 \) and in its vicinity one can reduce \( \sigma \) to canonical form

\[
\sigma = x_1 \, dx_1 \wedge dx_2 + dx_3 \wedge dx_4 \tag{7}
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$$\sigma = x_1 dx_1 \wedge dx_2 + dx_3 \wedge dx_4 \quad (7)$$

while in the vicinity of $\bar{x} \in \Lambda$ canonical form is

$$\sigma = dx_1 \wedge dx_2 - x_4 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_4 + x_3 dx_2 \wedge dx_3 + x_4 dx_2 \wedge dx_4 + 2(x_1 - \frac{1}{2}(x_3^2 + x_4^2)) dx_3 \wedge dx_4 \quad (8)$$

(R. Roussarie, modified by $x_2 \mapsto x_2 - \frac{1}{2}x_3 x_4$).
Magnetic lines are described by

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As \( d = 2 \) and \( \sigma \) is defined by (6) magnetic line is a straight line \( \{x_1 = 0\} \). As \( d = 4 \) and \( \sigma \) is defined by (7) magnetic lines are straight lines \( \{x_1 = 0, x_3 = \text{const}, x_4 = \text{const}\} \). As \( d = 4 \) and \( \sigma \) is defined by (8) \( \Lambda = \{x_1 = x_3 = x_4 = 0\} \) and magnetic lines are helices \( \{x_1 = 0, x_3 = r \cos \theta, x_4 = r \sin \theta, x_2 = \text{const} - r^2 \theta / 2\} \) (with \( r = \text{const} \)) winging around \( \Lambda \).
True geometry

From the point of view of operator $H$ simultaneous analysis of form $\sigma$ and metrics $(g^{lj})$ is crucial. In particular eigenvalues $\pm if_j$ and eigenspaces of matrix $(F^l_k) = (\sum_j g^{lj} F_{jk})$ are really important.
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$$f_1 = F_{12}/\sqrt{g}, \quad g = \det(g^{jk})^{-1}$$

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$$ f_1 = F_{12}/\sqrt{g}, \quad g = \det(g^{jk})^{-1} \quad (10) $$

while for $d = 3$

$$ f_1 = \frac{1}{2} \left( \sum_{j,k,l,m} g^{jk} g^{lm} F_{jl} F_{km} \right)^{1/2} = \left( \sum_{j,k,l,m} g_{jk} F_j F_k \right)^{1/2} \quad (11) $$

where $F^j = \frac{1}{2} \sum_{k,l} \varepsilon^{jkl} F_{kl}$ is a vector intensity of magnetic field, $\varepsilon^{jkl}$ is an absolutely skew-symmetric tensor with $\varepsilon^{123} = 1/\sqrt{g}$. 
Assume first that $g^{jk}$, $F_{jk}$ and $V$ are constant. Then with no loss of the generality one can assume that $g^{jk} = \delta_{jk}$, skew-symmetric matrix $(F_{jk})$ is reduced to the canonical form:

$$F_{jk} = \begin{cases} 
  f_j & j = 1, \ldots, r, \ k = j + r \\
  -f_k & j = r + 1, \ldots, 2r, \ k = j - r \\
  0 & \text{otherwise}
\end{cases}$$

(12)

$f_j > 0$ and $V = 0$. 

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Then as $d = 2$, $f_1 > 0$ a classical particle described by Hamiltonian

$$H(x, \xi) = \frac{1}{2} \left( \sum_{j,k} g^{jk}(x) (\xi_j - \mu A_j(x)) (\xi_k - \mu A_k(x)) - V(x) \right)$$  \hspace{1cm} (13)
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moves along cyclotrons which in this case are circles of radius $\rho_1 = (\mu f_1)^{-1} \sqrt{2E}$ with the angular velocity $\omega_1 = \mu f_1$ on the energy level $\{H(x, \xi) = E\}$. 
As $d = 3$, $f_1 > 0$ there are a cyclotron movement along circles of radii $\rho_1 = (\mu f_1)^{-1} \sqrt{2E_1}$ with the angular velocity $\omega_1 = \mu f_1$.
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$$\rho_1 = (\mu f_1)^{-1} \sqrt{2E_1}$$

with the angular velocity $\omega_1 = \mu f_1$ and a free movement along magnetic lines (which are straight lines along Ker $F$) with a speed $\sqrt{2E_f}$ and energy $E$ is split into two constant arbitrary parts $E = E_1 + E_f$. 

**Figure:** 3D “constant” case
Multidimensional case with $d = 2r = \text{rank } F$ is a combination of 2D cases: there are $r$ cyclotron movements with angular velocities $\omega_k = \mu f_k$ and radii $\rho_k = (\mu f_k)^{-1} \sqrt{2E_k}$.
Multidimensional case with $d = 2r = \text{rank } F$ is a combination of 2D cases: there are $r$ cyclotron movements with angular velocities $\omega_k = \mu f_k$ and radii $\rho_k = (\mu f_k)^{-1} \sqrt{2E_k}$ where energy $E$ is split into $r$ constant arbitrary parts $E = E_1 + E_2 + \cdots + E_r$. 
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As $d > 2r = \text{rank } F$ in addition to the cyclotronic movements described above appears a free movement along any constant direction $\vec{v} \in \text{Ker } F$ with a speed $\sqrt{2E_f}$.
Multidimensional case with \( d = 2r = \text{rank } F \) is a combination of 2D cases: there are \( r \) cyclotron movements with angular velocities \( \omega_k = \mu f_k \) and radii \( \rho_k = (\mu f_k)^{-1} \sqrt{2E_k} \) where energy \( E \) is split into \( r \) constant arbitrary parts \( E = E_1 + E_2 + \cdots + E_r \). The exact nature of the trajectories depends on the comeasurability of \( f_1, \ldots, f_r \).

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Multidimensional case

Multidimensional case with $d = 2r = \text{rank } F$ is a combination of 2D cases: there are $r$ cyclotron movements with angular velocities $\omega_k = \mu f_k$ and radii $\rho_k = (\mu f_k)^{-1} \sqrt{2E_k}$ where energy $E$ is split into $r$ constant arbitrary parts $E = E_1 + E_2 + \cdots + E_r$. The exact nature of the trajectories depends on the comeasurability of $f_1, \ldots, f_r$.

As $d > 2r = \text{rank } F$ in addition to the cyclotronic movements described above appears a free movement along any constant direction $\vec{v} \in \text{Ker } F$ with a speed $\sqrt{2E_f}$ where energy $E$ is split into $r + 1$ constant arbitrary parts $E = E_1 + E_2 + \cdots + E_r + E_f$.

This difference between cases $d = 2r = \text{rank } F$ and $d > 2r = \text{rank } F$ will be traced through the whole talk.
Assume now only that $d = \text{rank } F$. 
Magnetic Drift

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Magnetic Drift

Assume now only that \( d = \text{rank } F \). In addition assume temporarily that \( F_{jk} \) and \( g^{jk} \) are constant but \( V(x) \) is linear.

Then cyclotron movement(s) is combined with the magnetic drift described by equation

\[
\frac{dx_j}{dt} = (2\mu)^{-1} \sum_k \Phi^{jk} \partial_k V
\]  

(14)

where \( (\Phi^{jk}) = (F_{jk})^{-1} \).
As $d = 2$ it will be movement along cycloid

**Figure:** Cycloid
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and multidimensional movement will be combination of those.
Not assuming anymore that $V$ is linear we get a bit more complicated picture:
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- Equation (14) holds modulo $O(\mu^{-2})$; modulo error $O(\mu^{-2}t)$
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- Equation (14) holds modulo $O(\mu^{-2})$; modulo error $O(\mu^{-2} t)$
- As $d = 2$ cycloid is replaced by a more complicated curve drifting along $V = \text{const}$ and thus cyclotron radius $\rho = (\mu f_1)^{-1} \sqrt{2E + V}$ would be preserved.
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- Equation (14) holds modulo $O(\mu^{-2})$; modulo error $O(\mu^{-2} t)$
- As $d = 2$ cycloid is replaced by a more complicated curve drifting along $V = \text{const}$ and thus cyclotron radius $\rho = (\mu f_1)^{-1} \sqrt{2E + V}$ would be preserved.
- In higher dimensions all cyclotron radii are preserved as well.
Without assumption that $g^{jk}$ and $F_{jk}$ are constant picture becomes even more complicated:
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- As $d = 2$ cycloid is replaced by a more complicated curve drifting along $f^{-1}(V + 2E) = \text{const}$ (thus preserving angular momentum $\omega_1 \rho_1^2$ according to equation

$$\frac{dx}{dt} = (2\mu)^{-1}(\nabla f^{-1}(V + 2E))\perp$$

(15)

where $\perp$ means clockwise rotation by $\pi/2$ assuming that at point in question $g^{jk} = \delta_{jk}$. 

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- As $d = 2$ cycloid is replaced by a more complicated curve drifting along $f^{-1}(V + 2E) = \text{const}$ (thus preserving angular momentum $\omega_1 \rho_2^2$ according to equation

$$\frac{dx}{dt} = (2\mu)^{-1}(\nabla f^{-1}(V + 2E)) \perp$$

where $\perp$ means clockwise rotation by $\pi/2$ assuming that at point in question $g^{jk} = \delta^{jk}$.

- In higher dimensions (at least as non-resonance conditions $f_j \neq f_k \ \forall j \neq k$ and $f_j \neq f_k + f_l \ \forall j, k, l$ are fulfilled) one can split potential $V = V_1 + \cdots + V_k$ so that similar equations hold in each eigenspace of $(F_j^l)^2$ and both separate energies and angular momenta are (almost) preserved.
3D case

As $d > 2r = \text{rank } F$ the free movement is the main source of the spatial displacement and the most interesting case is $2r = d - 1$ and especially $d = 3, r = 1$. 
3D case

As \( d > 2r = \text{rank } F \) the free movement is the main source of the spatial displacement and the most interesting case is \( 2r = d - 1 \) and especially \( d = 3, \ r = 1. \)
In this case the magnetic angular momentum \( M \) is (almost) preserved;
As \( d > 2r = \text{rank } F \) the free movement is the main source of the spatial displacement and the most interesting case is \( 2r = d - 1 \) and especially \( d = 3, \ r = 1 \).
In this case the magnetic angular momentum \( M \) is (almost) preserved; thus \textbf{kinetic energy} of magnetic rotation is \( \frac{1}{2} f^{-1} M^2 \);
As $d > 2r = \text{rank } F$ the free movement is the main source of the spatial displacement and the most interesting case is $2r = d - 1$ and especially $d = 3$, $r = 1$.

In this case the magnetic angular momentum $M$ is (almost) preserved; thus kinetic energy of magnetic rotation is $\frac{1}{2} f^{-1} M^2$; therefore in the coordinate system such that $g^{1j} = \delta_{1j}$ the free movement is described by 1D Hamiltonian

$$H_1(x_1, \xi_1; x', M) = \frac{1}{2} \xi_1^2 - \frac{1}{2} V_{\text{eff}}$$

with effective potential $V_{\text{eff}}(x_1, x') = V - f^{-1} M^2$, $x = (x_1, x')$. 
Thus particle does not necessarily run the whole magnetic line and the helix winging around it does not necessarily have constant step or radius.

**Figure**: 3D Movement in the constant electric field along magnetic line
Thus particle does not necessarily run the whole magnetic line and the helix winging around it does not necessarily have constant step or radius. Effect of magnetic drift is rather minor.

**Figure:** 3D Movement in the constant electric field along magnetic line

**Figure:** 3D Movement in the constant electric field orthogonal to magnetic line
2D case: variable rank

Situation becomes really complicated for variable rank $F$. I am going to consider only $d = 2, 4$ and a generic magnetic form $\sigma$. 

\[ H_0 = \frac{1}{2} \left( \xi_{1}^2 + \left( \xi_{2}^2 - \mu x_1/\nu \right)^2 - 1 \right) \]  

the drift equation is 
\[ dx_1/dt = 0, \quad dx_2/dt = 2(\nu - 1)\mu - 1 x - \nu \]  

and for $|x_1| \gg \bar{\gamma} = \mu - 1/\nu$ gives a proper description of the picture.
Situation becomes really complicated for variable rank $F$. I am going to consider only $d = 2, 4$ and a generic magnetic form $\sigma$. I start from the model Hamiltonian as $d = 2$:

$$H^0 = \frac{1}{2} \left( \xi_1^2 + (\xi_2 - \mu x_1^{\nu}/\nu)^2 - 1 \right)$$  \hspace{1cm} (17)

the drift equation is

$$\frac{dx_1}{dt} = 0, \quad \frac{dx_2}{dt} = \frac{1}{2} (\nu - 1) \mu^{-1} x_1^{-\nu}$$  \hspace{1cm} (18)

and for $|x_1| \gg \bar{\gamma} = \mu^{-1/\nu}$ gives a proper description of the picture.
For model Hamiltonian (17) with $\mu = 1$ (otherwise one can scale $x_1 \mapsto \mu^{1/\nu} x_1$, $\xi_2 \mapsto \mu k$)
For model Hamiltonian (17) with \( \mu = 1 \) (otherwise one can scale \( x_1 \mapsto \mu^{1/\nu} x_1 \), \( \xi_2 \mapsto \mu k \)) we can consider also 1-dimensional movement along \( x_1 \) with potential

\[
\mathcal{V}(x_1; k) = 1 - (k - x_1^{\nu}/\nu)^2, \quad k = \xi_2; \quad (19)
\]
Then for odd $\nu$

- We have one-well potential;

**Figure**: Odd $\nu$, $k > 1$;  

**Figure**: Odd $\nu$, $0 < k < 1$. 

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As $k = \pm 1$ one of its extremes is 0 where $\frac{d\nu}{dx_1}(0) = 0$;

**Figure:** Odd $\nu$, $k = 1$;
As \( k = \pm 1 \) one of its extremes is 0 where \( \frac{d\nu}{dx_1}(0) = 0 \);

Well is more to the right/left from 0 as \( \pm k > 0 \); as \( k = 0 \) well becomes symmetric.

**Figure:** Odd \( \nu \), \( k = 1 \);

**Figure:** Odd \( \nu \), \( k = 0 \).
For even \( \nu \) potential is always symmetric and
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**Figure:** Even $\nu$, $k > 1$;
For even $\nu$ potential is always symmetric and

- We have two-well potential with the central bump above surface if $k > 1$
- and below it as $0 < k < 1$:

**Figure:** Even $\nu$, $k > 1$;

**Figure:** Even $\nu$, $0 < k < 1$. 
And exactly on the surface if $k = 1$;

**Figure:** Even $\nu$, $k = 1$;
And exactly on the surface if $k = 1$;

There is no central bump as $-1 < k < 0$;

**Figure:** Even $\nu$, $k = 1$;

**Figure:** Even $\nu$, $-1 < k < 0$. 

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- And exactly on the surface if $k = 1$;
- There is no central bump as $-1 < k < 0$;
- For $k \leq -1$ the well disappears.

**Figure:** Even $\nu$, $k = 1$;  
**Figure:** Even $\nu$, $-1 < k < 0$. 

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Consider trajectories on the energy level 0.
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but one needs to analyze the increment of \( x_2 \) during this period

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I(k) = 2 \int_{x_1^-(k)}^{x_1^+(k)} \frac{(k - x_1''/\nu)dx_1}{\sqrt{2V(x_1; k)}}.
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One can prove that $I(k) \geq 0$ as $k \geq k^*$ with $0 < k^* < 1$ for even $\nu$ and $k^* = 0$ for odd $\nu$. In particular, $k^* \approx 0.65$ for $\nu = 2$. 

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Magnetic Schrödinger Operator
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\]

One can prove that \( I(k) \gtrless 0 \) as \( k \gtrless k^* \) with \( 0 < k^* < 1 \) for even \( \nu \) and \( k^* = 0 \) for odd \( \nu \). In particular, \( k^* \approx 0.65 \) for \( \nu = 2 \). Further, \( I(k) \propto (k - k^*) \) as \( k \approx k^* \).
Trajectories on \((x_1, x_2)\)-plane plotted by Maple for even \(\nu\):

**Figure:** \(k \gg 1\); as \(x_1 > 0\) trajectory moves up and rotates clockwise
Trajectories on \((x_1, x_2)\)-plane plotted by Maple for even \(\nu\):

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**Figure:** \(k\) decreases, still \(k > 1\). Trajectory becomes less tight; actual size of cyclotrons increases and the drift is faster
**Figure:** $k$ further decreases, still $k > 1$. Trajectory becomes even less tight; actual size of cyclotrons increases and the drift is faster.
\textbf{Figure}: \(k\) further decreases, still \(k > 1\). Trajectory becomes even less tight; actual size of cyclotrons increases and the drift is faster.

\textbf{Figure}: \(k = 1\). Trajectory contains just one cyclotron.
**Figure:** $k$ further decreases, still $k > 1$. Trajectory becomes even less tight; actual size of cyclotrons increases and the drift is faster.

These trajectories have mirror-symmetric as $x_1 < 0$ with movement up and counter-clockwise.

**Figure:** $k = 1$. Trajectory contains just one cyclotron.
As $k < 1$ we cover both positive and negative $x_1$:
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**Figure:** $k$ further decays but still larger than $k^*$. Drift slows down
**Figure:** $k = k^*$. No drift; trajectory becomes periodic.
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**Figure:** $k < k^*$. Drift now is down!
**Figure:** $k$ decays further. Drift down accelerates.
**Figure:** $k$ decays further. Drift down accelerates.

**Figure:** and further; as $k = -1$ we have just straight line down.
Consider odd $\nu$ now.
Consider odd $\nu$ now. We need to consider $k \geq 0$ only because for $k < 0$ picture will be obtained by the central symmetry.
Consider odd ν now. We need to consider $k \geq 0$ only because for $k < 0$ picture will be obtained by the central symmetry. Also as $k \geq 1$ pictures for odd and even ν look similar since $x_1$ is positive along trajectories.
Consider odd \( \nu \) now. We need to consider \( k \geq 0 \) only because for 
\( k < 0 \) picture will be obtained by the central symmetry. 
Also as \( k \geq 1 \) pictures for odd and even \( \nu \) look similar since \( x_1 \) is positive along trajectories. 
So, consider \( 0 \leq k < 1 \).
**Figure:** $k < 1$ slightly. Drift is up and the fastest.
**Figure:** $k < 1$ slightly. Drift is up and the fastest

**Figure:** $k$ decays. Drift up slows down
**Figure:** $k$ decays further. Drift up slows down further.
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For the spectral asymptotics periodic trajectories are very important, especially short ones.
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\textbf{Figure: } $\nu$ is even
For the spectral asymptotics periodic trajectories are very important, especially short ones. Periodic trajectories shown above are very unstable and taking $V = 1 - \alpha x_1$ instead of $x_1$ breaks them down.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{even.png}
\caption{$\nu$ is even}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{odd.png}
\caption{$\nu$ is odd}
\end{figure}
The most natural model operator corresponding to the canonical form

\[ \sigma = x_1 dx_1 \wedge dx_2 + dx_3 \wedge dx_4 \]  

is \( H^0 + H'' \) with \( H^0 \) as above and \( 2H'' = \xi_3^2 + (\xi_4 - x_3)^2 \).
The most natural model operator corresponding to the canonical form

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is \( H^0 + H'' \) with \( H^0 \) as above and \( 2H'' = \xi_3^2 + (\xi_4 - x_3)^2 \). Then \( H'' \) is a movement integral.
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is $H^0 + H''$ with $H^0$ as above and $2H'' = \xi_3^2 + (\xi_4 - x_3)^2$. Then $H''$ is a movement integral. Therefore the dynamics is split into dynamics in $(x', \xi') = (x_1, x_2, \xi_1, \xi_2)$ described above with potential $W = V - 2E$ and standard cyclotron movement with energy $E$ in $(x'', \xi'') = (x_3, x_4, \xi_3, \xi_4)$. 
The most natural model operator corresponding to the canonical form

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Situation actually is way more complicated: considering \( H^0 + (1 + \alpha x_1)E \) we arrive to the 1-D potential \( V - 1(1 + \alpha x_1) \) and playing with \( E \) and \( \alpha \) one can kill the drift even for \( k \gg 1 \) leading to many periodic trajectories.
Consider canonical form (8) which in polar coordinates in \((x_3, x_4)\) becomes

\[
\sigma = d \left( (x_1 - \frac{1}{2} \rho^2) dx_2 + (x_1 - \frac{1}{4} \rho^2) \rho^2 d\theta \right).
\]  
(22)
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(22)

The most natural classical Hamiltonian corresponding to this form is

\[
2H = \xi_1^2 + \left( \xi_2 - \mu (x_1 - \frac{1}{2} \rho^2) \right)^2 + \\
\varrho^2 + r^{-2} \left( \vartheta - \mu (x_1 - \frac{1}{4} \rho^2) \rho^2 \right)^2 - 1
\]

(23)

with \(\varrho, \vartheta\) dual to \(\rho, \theta\).
Consider canonical form (8) which in polar coordinates in \((x_3, x_4)\) becomes

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The most natural classical Hamiltonian corresponding to this form is

\[
2H = \xi_1^2 + (\xi_2 - \mu(x_1 - \frac{1}{2} \rho^2))^2 + \\
\varrho^2 + r^{-2} (\vartheta - \mu(x_1 - \frac{1}{4} \rho^2) \rho^2)^2 - 1
\]

(23)

with \(\varrho, \vartheta\) dual to \(\rho, \theta\).

Note that \(\xi_2\) and \(\vartheta\) are movement integrals and therefore \(x_1 - \frac{1}{2} \rho^2\) is preserved modulo \(O(\mu^{-1})\).
Based on this one can prove that

- There is a cyclotronic movement with the angular velocity \( \propto \mu^{-1} \) in the normal direction to paraboloid
  \[ \{-x_1 + \frac{1}{2}\rho^2 = \frac{1}{2}\bar{\rho}^2\}; \]
Based on this one can prove that

- There is a cyclotronic movement with the angular velocity $\sim \mu^{-1}$ in the normal direction to parabloid $\{ -x_1 + \frac{1}{2} \rho^2 = \frac{1}{2} \bar{\rho}^2 \}$;

- combined in the zone $\{|x_1| \leq c\rho^2 \}$ with the movement similar to one described in 2D case in $(\rho, \theta)$-coordinates (with $\{x_1 = 0\}$ now equivalent to $\{\rho = \bar{\rho}\}$) on the surface of this parabloid;
Based on this one can prove that

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- and also combined some movement along \( x_2 \);
Based on this one can prove that

- There is a cyclotronic movement with the angular velocity \( \approx \mu^{-1} \) in the normal direction to paraboloid \( \{-x_1 + \frac{1}{2}\rho^2 = \frac{1}{2}\bar{\rho}^2\}; \)

- combined in the zone \( \{|x_1| \leq c\rho^2\} \) with the movement similar to one described in 2D case in \((\rho, \theta)\)-coordinates (with \(\{x_1 = 0\} \) now equivalent to \(\{\rho = \bar{\rho}\}\)) on the surface of this paraboloid;

- and also combined some movement along \(x_2\);

- I did not consider zone \( \{|x_1| \geq c\rho^2\} \) since it was not needed for the spectral asymptotics.
In the case $d = 2$ and a full-rank magnetic field microlocal canonical form (Birkhoff normal form) of Magnetic Schrödinger operator is (1/2 of)

\[
\omega_1(x_1, \mu^{-1} hD_1)(h^2 D_2^2 + \mu^2 x_2^2) - W(x_1, \mu^{-1} hD_1) + \sum_{m+k+l \geq 2} a_{mkl}(x_1, \mu^{-1} hD_1)(h^2 D_2^2 + \mu^2 x_2^2)^m \mu^{2-2m-2k-l} h^l \quad (24)
\]
In the case $d = 2$ and a full-rank magnetic field microlocal canonical form (Birkhoff normal form) of Magnetic Schrödinger operator is ($\frac{1}{2}$ of)

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with $\omega_j = f_j \circ \Psi$, $W = V \circ \Psi$ with some map $\Psi$. 
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with $\omega_j = f_j \circ \Psi$, $W = V \circ \Psi$ with some map $\Psi$. The first line is main part of the canonical form.
In the case $d = 3$ and a maximal-rank magnetic field microlocal canonical form (Birkhoff normal form) of Magnetic Schrödinger operator is $(\frac{1}{2})$ of

$$\omega_1(x_1, x_2, \mu^{-1}hD_2)(h^2D_3^2 + \mu^2x_3^2) + h^2D_1^2 - W(x_1, x_2, \mu^{-1}hD_2) +$$

$$\sum_{m+n+k+l \geq 2} a_{mnkl}(x_1, x_2, \mu^{-1}hD_2)(h^2D_3^2 + \mu^2x_3^2)^mD_1^n \times$$

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In the case $d = 3$ and a maximal-rank magnetic field microlocal canonical form (Birkhoff normal form) of Magnetic Schrödinger operator is \((\frac{1}{2})\) of

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\sum_{m+n+k+l \geq 2} a_{mnkl}(x_1, x_2, \mu^{-1}hD_2)(h^2D_3^2 + \mu^2x_3^2)^mD_1^n \times \\
\mu^{2-2m-2k-l-n}h^{l+n} \tag{25}
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In the case \( d \geq 4 \) and a constant rank magnetic field microlocal canonical form (\textit{Birkhoff normal form}) of Magnetic Schrödinger operator is of the similar type provided we can avoid some obstacles:
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If $f_j$ have constant multiplicities (say are simple for simplicity) then the main part is

$$
\sum_{1 \leq j \leq r} \omega_j(x', x'', \mu^{-1} hD'')(h^2 D_{r+q+j}^2 + \mu^2 x_{r+q+j}^2) + h^2 D'^2 - W(x', x'', \mu^{-1} hD''); \quad (26)
$$

where $x' = (x_1, \ldots, x_q)$, $x'' = (x_{q+1}, \ldots, x_{q+r})$, $2r = \text{rank } F$, $q = d - 2r$. 

Victor Ivrii
Magnetic Schrödinger Operator
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where $x' = (x_1, \ldots, x_q)$, $x'' = (x_{q+1}, \ldots, x_{q+r})$, $2r = \text{rank } F$, $q = d - 2r$.

Next terms appear if we can avoid higher order resonances: $\sum_j p_j f_j(x) = 0$ with $p_j \in \mathbb{Z}$ and $3 \leq \sum_j |p_j|$ order of the resonance.
After operator reduced to canonical form one can decompose functions as

\[ u(x) = \sum_{\alpha \in \mathbb{Z}^+^r} u_{\alpha}(x', x'') \Upsilon_{p_1}(x_{r+q+1}) \cdots \Upsilon_{p_r}(x_d) \quad (27) \]

where \( \Upsilon \) are eigenfunctions of Harmonic oscillator \( h^2D^2 + \mu^2x^2 \) (i.e. scaled Hermite functions).
After operator reduced to canonical form one can decompose functions as

\[ u(x) = \sum_{\alpha \in \mathbb{Z}^{+r}} u_{\alpha}(x', x'') \Upsilon_{p_1}(x_{r+q+1}) \cdots \Upsilon_{p_r}(x_d) \]  

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where \( \Upsilon \) are eigenfunctions of Harmonic oscillator \( h^2D^2 + \mu^2x^2 \) (i.e. scaled Hermite functions).

Then as \( 2r = d \) we get a family of \( r \)-dimensional \( \mu^{-1}h \)-PDOs.
After operator reduced to canonical form one can decompose functions as

\[ u(x) = \sum_{\alpha \in \mathbb{Z}^{+r}} u_{\alpha}(x', x'') \gamma_{p_1}(x_{r+q+1}) \cdots \gamma_{p_r}(x_d) \]  

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where \( \gamma \) are eigenfunctions of Harmonic oscillator \( \hbar^2 D^2 + \mu^2 x^2 \) (i.e. scaled Hermite functions).

Then as \( 2r = d \) we get a family of \( r \)-dimensional \( \mu^{-1} \hbar \)-PDOs and for \( 2r < d \) we get a family of \( q \)-dimensional Schrödinger operators with potentials which are \( r \)-dimensional \( \mu^{-1} \hbar \)-PDOs.
The similar approach works for 2 and 4-dimensional Schrödinger operators with degenerate magnetic field of the types we considered before but only in the far outer zone
\[ \{ \gamma(x) \overset{\text{def}}{=} |x_1| \gg \mu^{-1/\nu} \} \] and to this form operator is reduced in balls \( B(\bar{x}, \frac{1}{2} \gamma(\bar{x})) \).
As $d = 2, 4$ there is also a more global canonical form.
As $d = 2, 4$ there is also a more global canonical form. As $d = 2$ this form is (after multiplication by some non-vanishing function) in zone $\{|x_1| \ll 1\}$

$$h^2 D_1^2 + (hD_2 - \mu x_1^\nu/\nu)^2 - W(x) + \text{perturbation} \quad (28)$$

with $W = V\phi^{-2/(\nu+1)}$ (if originally $f_1 \sim \phi \text{dist}(x, \Sigma)^{\nu-1}$, $\Sigma = \{x_1 = 0\}$).
For $d = 4$ we can separate cyclotron part corresponding to non-vanishing eigenvalue $f_2$ after what we get a 3-dimensional second-order DO (+ perturbation) with the principal part which is the quadratic form of rank 2 and a free term $V - (2\alpha + 1)\mu hf_2$ where $\alpha \in \mathbb{Z}^+$ is a corresponding magnetic quantum number.
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This operator could be reduced to the form similar to (28) (at least away from $\Lambda = \{x_3 = x_4 = 0\}$); here $W = (V - (2\alpha + 1)\mu hf_2)\phi^{-2/3}$. 
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This operator could be reduced to the form similar to (28) (at least away from $\Lambda = \{x_3 = x_4 = 0\}$); here

$W = (V - (2\alpha + 1)\mu hf_2)\phi^{-2/3}$.

Close to $\Lambda$ but as $|x_1| \leq C\rho^2$ we can get a similar form but with $\theta$ instead of $x_2$ and $\mu \rho$ instead of $\mu$. 
One can prove that semiclassical quantum dynamics follows the classical long enough to recover sharp remainder estimates but the notion of periodic orbit should be adjusted to reflect (logarithmic) uncertainty principle.

\[ |\text{osc}(x)\| - |\text{osc}(\hbar D)| \geq C\hbar \log\hbar \]
One can prove that semiclassical quantum dynamics follows the classical long enough to recover sharp remainder estimates but the notion of periodic orbit should be adjusted to reflect \((\text{logarithmic})\) uncertainty principle

\[ |\text{osc}(x)| \cdot |\text{osc}(\hbar D)| \geq C\hbar|\log\hbar| \tag{29} \]

where \(\hbar\) is effective Plank constant (it could be \(h\) or \(\mu^{-1}h\) or one of them scaled depending on the particular situation).
One can prove that semiclassical quantum dynamics follows the classical long enough to recover sharp remainder estimates but the notion of periodic orbit should be adjusted to reflect (logarithmic) uncertainty principle

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where \(\hbar\) is effective Plank constant (it could be \(h\) or \(\mu^{-1}h\) or one of them scaled depending on the particular situation). We need logarithm because we are interesting in the size of the box outside of which function is negligible rather than in the mean quadratic deviation.
One can prove that semiclassical quantum dynamics follows the classical long enough to recover sharp remainder estimates but the notion of periodic orbit should be adjusted to reflect (logarithmic) uncertainty principle

$$|\text{osc}(x)| \cdot |\text{osc}(\hbar D)| \geq C\hbar|\log \hbar|$$  \hspace{1cm} (29)

where $\hbar$ is effective Plank constant (it could be $h$ or $\mu^{-1}h$ or one of them scaled depending on the particular situation). We need logarithm because we are interesting in the size of the box outside of which function is negligible rather than in the mean quadratic deviation. Function $\exp(-|x|^2/2\hbar)$ scaled shows that boxing requires logarithmic factor.
So instead of individual trajectories we consider their beams satisfying logarithmic uncertainty principle.

**Figure:** Classical and Semiclassical

We see that classical trajectory is not periodic but cannot say this about semiclassical beam
So instead of individual trajectories we consider their beams satisfying logarithmic uncertainty principle.

**Figure**: Classical and Semiclassical

We see that classical trajectory is not periodic but cannot say this about semiclassical beam until much larger time
I am looking at asymptotics as $h \to +0$, $\mu \to +\infty$ of

$$\Gamma(Qe)(0) = \int (e(\ldots, 0) Q^t_y)_{x=y} dy = \text{Tr}(Q E(0))$$

(30)

where $e(x, y, \tau)$ is the Schwartz kernel of the spectral projector $E(\tau)$ of $H$ and $Q$ is a pseudo-differential operator, $Q^t$ means a dual operator.
I am looking at asymptotics as \( h \to +0, \mu \to +\infty \) of

\[
\Gamma(Qe)(0) = \int (e(.,.,0)Q^t_y)_{x=y} dy = \text{Tr}(QE(0)) \tag{30}
\]

where \( e(x, y, \tau) \) is the Schwartz kernel of the spectral projector \( E(\tau) \) of \( H \) and \( Q \) is a pseudo-differential operator, \( Q^t \) means a dual operator.

As \( Q = I \) we get \( \text{Tr} E(0) \) which is the number of negative eigenvalues of \( H \) (and \( +\infty \) if there is an essential spectrum of \( H \) below 0).
I am looking at asymptotics as $h \to +0$, $\mu \to +\infty$ of

$$\Gamma(Qe)(0) = \int (e(.,.,0)Q^t_y)_{x=y} dy = \text{Tr}(QE(0)) \quad (30)$$

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As $Q = I$ we get $\text{Tr} E(0)$ which is the number of negative eigenvalues of $H$ (and $+\infty$ if there is an essential spectrum of $H$ below 0).

I hope to construct this expression (30) with $Q = I$ from itself for elements of the partition of unity with $Q = Q^* \geq 0$. 
Tauberian method Fourier says that the main part of $\Gamma(eQ_y^t)$ is given by

$$h^{-1} \int_{-\infty}^{0} \left( F_{t \to h^{-1}\tau} \bar{\chi}_{T}(t) \Gamma(uQ_y^t) \right) d\tau$$

(31)
Tauberian method Fourier says that the main part of $\Gamma(eQ^t_y)$ is given by

$$h^{-1} \int_{-\infty}^{0} \left( F_{t \to h^{-1} \tau} \tilde{\chi}_T(t) \Gamma(uQ^t_y) \right) d\tau$$

(31)

while remainder does not exceed $C \frac{M}{T} + C' h^s$ where

$$M = M_T = \sup_{|\tau| \leq \epsilon} \left| \left( F_{t \to h^{-1} \tau} \tilde{\chi}_T(t) \Gamma(uQ^t_y) \right) \right|$$

(32)

and $s$ is large, $C$ does not depend on $\epsilon, T, h, \mu$, $C'$ depends on $\epsilon > 0$. 
Here and below $u(x, y, t)$ is the Schwartz kernel of propagator $e^{ih^{-1}tH}$.
Here and below $u(x, y, t)$ is the Schwartz kernel of propagator $e^{ih^{-1}tH}$,

$\bar{\chi} \in C^\infty_0([−1, 1])$ equal 1 at $[−\frac{1}{2}, \frac{1}{2}]$,
$\chi \in C^\infty_0([−1, 1])$ equal 0 at $[−\frac{1}{2}, \frac{1}{2}]$, $\chi_T(t) = \chi(t/T), \ T > 0$
Here and below \( u(x, y, t) \) is the Schwartz kernel of propagator \( e^{ih^{-1}tH} \),

\[
\tilde{\chi} \in C_0^\infty([-1, 1]) \text{ equal 1 at } [-\frac{1}{2}, \frac{1}{2}], \\
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Actually remainder estimate persists if one replaces $T$ by any larger number $T'$ only in

$$h^{-1} \int_{-\infty}^{0} \left( F_{t \rightarrow h^{-1}T} \tilde{\chi}_{T'}(t) \Gamma(uQ^t_y) \right) d\tau$$  \hspace{1cm} (31)
Here and below $u(x, y, t)$ is the Schwartz kernel of propagator $e^{ih^{-1}tH}$,

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Actually remainder estimate persists if one replaces $T$ by any larger number $T'$ only in

\[ h^{-1} \int_{-\infty}^{0} \left( F_{t \to h^{-1}\tau} \bar{\chi} T'(t) \Gamma(uQ^t_y) \right) d\tau \]

So, we want to increase $T$ without (significantly) increasing $M_T$ in

\[ M = M_T = \sup_{|\tau| \leq \epsilon} \left| F_{t \to h^{-1}\tau} \bar{\chi} T(t) \Gamma(uQ^t_y) \right| \]
Evil of periodic trajectories

Microlocal analysis says that if there are no periodic trajectories with periods in \( \left[ \frac{T}{2}, T \right] \) on energy levels in \( [-2\epsilon, 2\epsilon] \) then

\[
\sup_{|\tau| \leq \epsilon} \left| \left( F_{t \rightarrow h^{-1}\tau} \chi_{\tau}(t) \Gamma(uQ^t_y) \right) \right| \leq C'h^s. \tag{33}
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Therefore if there are no periodic trajectories with periods in $[T, T']$ on energy levels in $[-2\epsilon, 2\epsilon]$, then one can retain $T$ in (31), $M_{T}$ in (32) but remainder estimate improves to $C \frac{M}{T'} + C'h^{s}$. 
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$$\sup_{|\tau| \leq \epsilon} \left| \left( F_{t \to h^{-1} \tau} x_{\tau}(t) \Gamma(uQ^t_y) \right) \right| \leq C' h^s. \quad (33)$$

Therefore if there are no periodic trajectories with periods in $[T, T']$ on energy levels in $[-2\epsilon, 2\epsilon]$, then one can retain $T$ in (31), $M_T$ in (32) but remainder estimate improves to $C \frac{M}{T'} + C' h^s$. So, periodic trajectories are one of the main obstacles in getting good remainder estimate. For example, if all trajectories are periodic with period $T = T_\Pi$ then it can happen that $M_T \asymp TT_\Pi^{-1} M_{T_\Pi}$ as $T \geq T_\Pi$ and increasing $T$ does not bring any improvement.
For example, let $\mu \leq 1$, $V \asymp 1$. Then there are no periodic trajectories with periods in $[T_0, T_1]$, $T_0 = Ch|\log h|$ and $T_1 = \epsilon$ because $\text{dist}(x(t), x(0)) \asymp T$ as $T \leq T_1$ and this distance is observable as $T \geq T_0$. 

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Also, taking $T$ really small in (31) allows us to calculate $u$ and (31) by a crude successive approximation method with unperturbed operator $H$ having coefficients frozen as $x = y$. 
As $\mu \geq 1$ the same arguments are true but $T_1 = \epsilon \mu^{-1}$ and the remainder estimate is $O(\mu h^{1-d})$. 

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Magnetic Schrödinger Operator
As $\mu \geq 1$ the same arguments are true but $T_1 = \epsilon \mu^{-1}$ and the remainder estimate is $O(\mu h^{1-d})$. This remainder estimate cannot be improved as $d = 2$, $g^{jk}$, $f_1$ and $V$ are constant.
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From direct calculations: as domain is $\mathbb{R}^2$ all eigenvalues are Landau levels $(\alpha + \frac{1}{2})\mu h - \frac{1}{2} V$ of infinite multiplicity ($\alpha \in \mathbb{Z}^+$) and

$$e(x, x, \tau) = \frac{1}{2\pi} \sum_{n \geq 0} \theta(2\tau + V - (2n + 1)\mu hf) \mu h^{-1} f \sqrt{g}$$

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with jumps $\sim \mu h^{-1}$ at Landau levels.
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But in many cases one can improve remainder estimate $O(\mu h^{1-d})$. 
From the point of view of applications one should take $Q$ with support (with respect to $x$) in ball $B(0, \frac{1}{2})$ (then rescaling arguments could be applied)
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So anything out of $B(0, 1)$ is a dark territory and we must take $T \leq T^*$ which is the time for which trajectory which started from supp $Q$ remains in $B(0, 1)$: as trajectory reaches the dark territory evil-doers living there can make it periodic with period slightly more than $T^*$. 
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So anything out of $B(0, 1)$ is a dark territory and we must take $T \leq T^*$ which is the time for which trajectory which started from $\text{supp } Q$ remains in $B(0, 1)$: as trajectory reaches the dark territory evil-doers living there can make it periodic with period slightly more than $T^*$.

But we can chose the time direction and we can chose it for every beam individually.

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Let \( d = 2 \) and \( f_1 \) do not vanish. Then
Now, as \( d = 3 \) the typical trajectory is non-periodic because of the free movement and

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- under certain non-degeneracy conditions breaking periodicity of the cyclotronic movement one can take $T_1 = T^*$ retaining $T_0 = Ch|\log h|$
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- under certain non-degeneracy conditions breaking periodicity of the cyclotronic movement one can take $T_1 = T^*$ retaining $T_0 = Ch|\log h|$

and the remainder estimate is $O(\mu^{-1}h^{1-d})$.

When tamed, our worst enemy (periodic trajectories) could become our best friend!
Theorem

Let $g^{jk}$, $F_{jk}$ and $V$ be constant and domain be $\mathbb{R}^d$. Then

$$e(x, x, E) = E_{d}^{MW}(x, E) \overset{\text{def}}{=} \Omega_{d-2r}(2\pi)^{-d+r} \mu^r h^{-d+r} \times$$

$$\sum_{\alpha \in \mathbb{Z}^+} \left(2E + V - \sum_{j}(2\alpha_j + 1)f_j \mu_{j}h\right)^{\frac{d-r}{2}} f_1 \cdots f_r \sqrt{g} \quad (35)$$

where $\Omega_k$ is a volume of unit ball in $\mathbb{R}^k$. 
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where $\Omega_{k}$ is a volume of unit ball in $\mathbb{R}^{k}$.

In particular, spectrum is pure point iff $r = d/2$. 
Results: $d = 2$

As $d = 2$ formula (35) provides a good approximation and non-degeneracy condition below breaks periodicity and provides a good remainder estimate:

**Theorem**

Let $d = 2$ and $g^{jk}$, $F_{jk}$, $V$ be smooth in $B(0,1)$, $f_1$ non-vanishing there and $\psi \in C_0^\infty(B(0,1))$. 

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**Theorem**

Let $d = 2$ and $g^{jk}, F_{jk}, V$ be smooth in $B(0,1)$, $f_1$ non-vanishing there and $\psi \in C_0^\infty(B(0,1))$. Let assume that all critical values of $V/f_1$ are non-degenerate. Then

$$| \int (e(x,x,0) - E_d^{MW}(x,0))\psi(x) \, dx | \leq C \mu^{-1} h^{1-d}$$

as $\mu \leq ch^{-1}$. 
Remark

- Estimate (36) holds in multidimensional full-rank case as well but non-degeneracy condition is pretty complicated;
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- As \( \epsilon_1 \leq \mu h \leq c \) nondegeneracy condition changes; as \( d = 2 \) it reads: \((2\alpha + 1)\mu h\) is not a degenerate critical value of \( V/f \) for any \( \alpha \in \mathbb{Z}^+ \);

- As \( f_1 + \cdots + f_r \geq \epsilon > 0 \) \( e(x, x, 0) \) is negligible and \( \mathcal{E}_d^{MW} = 0 \) for \( \mu \geq ch^{-1} \).
In this case remainder estimate cannot be better than $O(h^{1-d})$ but cannot be much worse:

**Theorem**

Let $g^{jk}$, $F_{jk}$, $V$ be smooth in $B(0, 1)$, $\text{rank}(F) = 2r$, $0 < 2r < d$ so $f_1, \ldots, f_r$ do not vanish there and $\psi \in C_0^\infty(B(0, 1))$. Then

- As either $2r = d - 1$ and some very mild non-degeneracy condition is fulfilled or $2r \leq d - 2$ or $\mu \leq h^{\delta-1}$ with $\delta > 0$

\[
| \int (e(x, x, 0) - \mathcal{E}_d^{MW}(x, 0)) \psi(x) \, dx | \leq Ch^{1-d}. \quad (37)
\]
Results: Non Full-rank Case

In this case remainder estimate cannot be better than $O(h^{1-d})$ but cannot be much worse:

Theorem

Let $g^{jk}$, $F_{jk}$, $V$ be smooth in $B(0,1)$, rank($F$) = $2r$, $0 < 2r < d$ so $f_1, \ldots, f_r$ do not vanish there and $\psi \in C_0^\infty(B(0,1))$. Then

- As either $2r = d - 1$ and some very mild non-degeneracy condition is fulfilled or $2r \leq d - 2$ or $\mu \leq h^{\delta-1}$ with $\delta > 0$

  $$| \int (e(x,x,0) - E_d^{MW}(x,0))\psi(x)\,dx | \leq C h^{1-d}. \quad (37)$$

- As $2r = d - 1$ left-hand expression does not exceed $C \mu h^{2-\delta-d} + C h^{1-d}$ with arbitrarily small $\delta > 0$. 

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Consider case $d = 2$, $f_1 \propto |x_1|^{\nu-1}$ with $\nu \geq 2$ assuming that

$$V \geq \epsilon_0 > 0. \quad (38)$$

We consider $\epsilon$-vicinity of $\{x_1 = 0\}$ with small enough constant $\epsilon > 0$. 
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Then in the outer zone $\{\tilde{\gamma} = C\mu^{-1/\nu} \leq |x_1| \leq \epsilon\}$ there is a drift with the speed $\mu^{-1}\gamma^{-\nu}$, this drift breaks periodicity and therefore contribution of the strip $\{|x_1| \asymp \gamma\}$ with $\gamma \in (\tilde{\gamma}, \epsilon)$ to the remainder estimate
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Then in the outer zone $\{\bar{\gamma} = C \mu^{-1/\nu} \leq |x_1| \leq \epsilon\}$ there is a drift with the speed $\mu^{-1} \gamma^{-\nu}$, this drift breaks periodicity and therefore contribution of the strip $\{|x_1| \approx \gamma\}$ with $\gamma \in (\bar{\gamma}, \epsilon)$ to the remainder estimate does not exceed $Ch^{1-d} \times \gamma \times \mu^{-1} \gamma^{-\nu}$ where the second factor is the width of the strip and the third one is the inverse “control time”. Then the total contribution of the outer zone to the remainder estimate does not exceed the same expression as $\gamma = \bar{\gamma}$ which is $C \mu^{-1/\nu} h^{1-d}$; this is our best shot.
In the inner zone \( \{|x_1| \leq \bar{\gamma}\} \) or equivalently \( \{|\xi_2| \leq C_0\} \) the similar arguments work as long as \( \rho \asymp |\xi_2 - k^* V^{1/2}| \geq \epsilon. \)
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the speed of drift is \( \asymp \rho \), the shift with respect to \( x_2 \) is \( \asymp \rho \tilde{\gamma} \) and in order to be observable it must satisfy logarithmic uncertainty principle \( \rho \tilde{\gamma} \times \rho \geq Ch|\log h| \) because characteristic scale in \( \xi_2 \) is \( \rho \) now. So, periodicity is broken provided

\[
\rho \geq \bar{\rho}_1 = C(\tilde{\gamma}^{-1} h|\log h|)^{1/2}
\]  

(40)
which leaves us with much smaller periodic zone

\[ Z_{\text{per}} = \left \{ |\xi_2 - k^* V^{1/2}| \leq \bar{\rho}_1 \right \}. \] (41)
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And in this periodic zone picking up \( T_1 \approx \bar{\gamma} \) we can pull remainder estimate \( C h^{-1} \bar{\rho}_1 \) which does not exceed our dream estimate \( C \mu^{-1/\nu} h^{-1} \) as \( \bar{\rho}_1 \leq \bar{\gamma} \) or

\[ \mu \leq C (h |\log h|)^{-\frac{\nu}{3}}. \]  

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And in this periodic zone picking up \( T_1 \simeq \bar{\gamma} \) we can pull remainder estimate \( Ch^{-1} \bar{\rho}_1 \) which does not exceed our dream estimate \( C \mu^{-1/\nu} h^{-1} \) as \( \bar{\rho}_1 \leq \bar{\gamma} \) or

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Actually for the general operator rather than model one we need to assume that \( \rho \geq C\bar{\gamma} \) but this does not spoil our dream estimate.
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Actually for the general operator rather than model one we need to assume that \( \rho \geq C \bar{\gamma} \) but this does not spoil our dream estimate. So we need to consider periodic zone defined by (41) assuming that (42) does not hold.
Inside of Periodic Zone

Even if in periodic zone $\mathcal{Z}_{\text{per}}$ periodicity of trajectories can be broken as $W = V\phi^{-1/\nu}|_{x_1=0}$ is “variable enough”:
Inside of Periodic Zone

Even if in periodic zone \( Z_{\text{per}} \) periodicity of trajectories can be broken as \( W = V \phi^{-1/\nu}|_{x_1=0} \) is “variable enough”:

**Theorem**

Let \( d = 2 \), \( f_1 = \phi(x) \text{dist}(x, \Sigma)^{\nu-1} \) with \( \nu \geq 2 \) and condition (38) be fulfilled. Then as \( \psi \in C_0^\infty(B(0, 1) \cap \{|x_1| \leq \epsilon\} \)

\[
| \int (e(x, x, 0) - E_d^{MW}(x, 0)) \psi(x) \, dx | \leq C(\mu^{-1/\nu} + \hbar^{(q+1)/2}) h^{1-d} \tag{43}
\]

where here and below \( \hbar = \mu^{1/\nu} h \), \( q = 0 \) in the general case and \( q = 1 \) under assumption \( W \) does not have degenerate critical points.
To improve this remainder estimate one should take into account the short periodic trajectories. Actually, periodicity of the trajectories close to them is broken but only after time $T_0 = C \rho^{-2} h|\log h|$. 
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$$\left| \int (e(x, x, 0) - \mathcal{E}_d^{MW}(x, 0)) \psi(x) \, dx - \int \mathcal{E}_{\text{corr}}^{MW}(x_2, 0) \psi(0, x_2) \, dx_2 \right| \leq C \mu^{-1/\nu} h^{1-d}$$

provided either very mild nondegeneracy condition is fulfilled or $\mu \leq ch^{\delta-\nu}$,
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**Theorem**

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provided either very mild nondegeneracy condition is fulfilled or \( \mu \leq c h^{\delta-\nu} \), where \( \mathcal{E}_{\text{corr}}^{MW}(x_2,0) \) is defined below.
Remark

In the case we are considering right now (and no other case considered in this talk) condition $f_1 + \cdots + f_r \geq \epsilon_0$ fails and therefore $e(x, x, 0)$ is not negligible as $\mu \geq ch^{-1}$;
**Remark**

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Remark

- In the case we are considering right now (and no other case considered in this talk) condition $f_1 + \cdots + f_r \geq \epsilon_0$ fails and therefore $e(x, x, 0)$ is not negligible as $\mu \geq ch^{-1}$;
- This happens only as $\mu \geq ch^{-\nu}$;
- As $ch^{-1} \leq \mu \leq ch^{-\nu}$ $E^{MW}(x, 0)$ is supported in $\{ |x_1| \leq \tilde{\gamma}_1 \stackrel{\text{def}}{=} C_0(\mu h)^{-1/(\nu-1)} \}$ where $\tilde{\gamma}_1 \geq \tilde{\gamma} = c\mu^{-1/\nu}$; therefore the main part of the spectral asymptotics (after integration) is of magnitude $(\mu h)^{-1/(\nu-1)} h^{-d}$.
In the theorem above

\[ E^\text{MW}_\text{corr}(x_2, \tau) = (2\pi \hbar)^{-1} \int n_0(\tau; x_2, \xi_2, \hbar) d\xi_2 - \int E^\text{MW}_0(\tau; x_1, x_2, \hbar) dx_1 \quad (45) \]
In the theorem above

\[ \mathcal{E}^{\text{MW}}(x_2, \tau) = (2\pi \hbar)^{-1} \int n_0(\tau; x_2, \xi_2, \hbar) \, d\xi_2 - \int \mathcal{E}_0^{\text{MW}}(\tau; x_1, x_2, \hbar) \, dx_1 \quad (45) \]

where \( n_0 \) is an eigenvalue counting function for an auxiliary 1D-operator

\[ a_0(x_2, \xi_2, \hbar) = \frac{1}{2} \left( \hbar^2 D_1^2 + (\xi_2 - x_1^\nu / \nu)^2 - W(x_2) \right) \quad (46) \]

and \( \mathcal{E}_0^{\text{MW}} \) is Magnetic Weyl approximation for related 2-dimensional operator.
Using Bohr-Sommerfeld approximation one can calculate eigenvalues of \( a_0(\xi_2) \) with \( O(\hbar^s) \) precision and \( E_{\text{MW corr}}^{\text{MW}}(x_2, \tau) \) with \( O(h^{-1}\hbar^s) \) precision.
Using Bohr-Sommerfeld approximation one can calculate eigenvalues of $a_0(\xi_2)$ with $O(\hbar^s)$ precision and $\mathcal{E}_{\text{MW}}^{\text{corr}}(x_2, \tau)$ with $O(h^{-1}\hbar^s)$ precision.

In particular modulo $O(h^{-1}\hbar) = O(\bar{\gamma}^{-1})$

$$\mathcal{E}_{\text{MW}}^{\text{corr}}(x_2, 0) \equiv \kappa h^{-1}\hbar^{\frac{1}{2}} W^{\frac{1}{4} - \frac{1}{4\nu}} G\left(\frac{S_0 W^{\frac{1}{2}} + \frac{1}{2\nu}}{2\pi\hbar}\right)$$

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with some constants $\kappa$ and $S_0$
Using Bohr-Sommerfeld approximation one can calculate eigenvalues of \( a_0(\xi_2) \) with \( O(\hbar^s) \) precision and \( \mathcal{E}_{\text{corr}}^{\text{MW}}(x_2, \tau) \) with \( O(h^{-1}\hbar^s) \) precision. In particular modulo \( O(h^{-1}\hbar) = O(\bar{\gamma}^{-1}) \)

\[
\mathcal{E}_{\text{corr}}^{\text{MW}}(x_2, 0) \equiv \tau h^{-1} \hbar^{\frac{1}{2}} W^{\frac{1}{4}} - \frac{1}{4\nu} G\left(\frac{S_0 W^{\frac{1}{2}} + 1}{2\pi \hbar}\right) \tag{47}
\]

with some constants \( \tau \) and \( S_0 \) and function \( G \) defined by

\[
G(t) = \int_{\mathbb{R}} \left( t + \frac{1}{2} \eta^2 - \left[ t + \frac{1}{2} \eta^2 + \frac{1}{2} \right] \right) d\eta \tag{48}
\]

with the converging integral in the right-hand expression.
One can prove easily that

\[ G \not\equiv 0, \quad G(t + 1) = G(t), \quad \int_{0}^{1} G(t) \, dt = 0, \quad G \in C^{\frac{1}{2}}. \]  

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\[ G \not\equiv 0, \quad G(t + 1) = G(t), \quad \int_0^1 G(t) \, dt = 0, \quad G \in C^{\frac{1}{2}}. \]  

This is one of examples of short periodic trajectories really contributing to asymptotics.
Results: Degenerating 4D case

4D case is way more complicated than 2D one. But there are some good news: since \( f_1 + f_2 \geq \epsilon \) we need to consider only \( \mu \leq ch^{-1} \).
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4D case is way more complicated than 2D one. But there are some good news: since $f_1 + f_2 \geq \epsilon$ we need to consider only $\mu \leq ch^{-1}$. The main difficulty in 4D case comes from outer zone $\{ \tilde{\gamma} = C\mu^{-1/\nu} \leq |x_1| \leq \epsilon \}$ because there could be short periodic trajectories.
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4D case is way more complicated than 2D one. But there are some good news: since $f_1 + f_2 \geq \epsilon$ we need to consider only $\mu \leq ch^{-1}$. The main difficulty in 4D case comes from outer zone $\{ \bar{\gamma} = C\mu^{-1/\nu} \leq |x_1| \leq \epsilon \}$ because there could be short periodic trajectories. In other words: Landau level $(2\alpha_1 + 1)\mu hf_1 + (2\alpha_2 + 1)\mu hf_2$ could be flat 0.
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$(2\alpha_1 + 1)\mu hf_1 + (2\alpha_2 + 1)\mu hf_2$ could be flat 0. Still in contrast to the very general case when this can happen for up to $(\mu h)^{-1}$ pairs $\alpha \in \mathbb{Z}^2$, in the assumptions of theorem below it can happen only for no more than $C$ pairs and I was able to prove
Let $F$ is of Martinet-Roussarie type and

$$V \geq \epsilon_0 > 0.$$  (38)
Theorem

Let $F$ is of Martinet-Roussarie type and

$$V \geq \epsilon_0 > 0.$$  \hspace{1cm} (38)

Then as $\psi$ is supported in $B(0, 1) \cap \{|x_1| \leq \epsilon\}$

$$|\int (e(x, x, 0) - E_{MW}^4(x, 0)) \psi(x) \, dx| \leq C \mu^{-1/2} h^{-3} + C \mu^2 h^{-2}. \hspace{1cm} (50)$$
One can improve this result under extra condition breaking flat Landau levels:
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**Theorem**

In frames of above theorem assume that $(V/f_2)_{x_1=0}$ does not have degenerate critical points.
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*In frames of above theorem assume that \((V/f_2)_{x_1=0}\) does not have degenerate critical points. Then as \(\psi\) is supported in \(B(0,1) \cap \{|x_1| \leq \epsilon\}\)*

\[
| \int (e(x,x,0) - E_{4MW}(x,0) - E_{corr}(x,0)) \psi(x) \, dx | \leq C\mu^{-1/2}h^{-3}. \tag{51}
\]
One can improve this result under extra condition breaking flat Landau levels:

**Theorem**

In frames of above theorem assume that \((V/f_2)_{x_1=0}\) does not have degenerate critical points. Then as \(\psi\) is supported in \(B(0, 1) \cap \{|x_1| \leq \epsilon\}\)

\[
| \int (e(x, x, 0) - \varepsilon_{4\text{MW}}(x, 0) - \varepsilon_{\text{corr}}^{\text{MW}}(x, 0)) \psi(x) \, dx | \leq C \mu^{-1/2} h^{-3}. \tag{51}
\]

Here \(\varepsilon_{\text{corr}}^{\text{MW}} = O(\mu^{5/4} h^{-3/2})\) is associated with periodic zone \(\{|x_1| \leq c\mu^{-1/2}\}\), and is the sum of similar expressions in 2D case for \(V_\beta = V - (2\beta + 1)\mu hf_2\) with \(\beta \in \mathbb{Z}^+\);
One can improve this result under extra condition breaking flat Landau levels:

**Theorem**

In frames of above theorem assume that \((V/f_2)_{x_1=0}\) does not have degenerate critical points. Then as \(\psi\) is supported in \(B(0, 1) \cap \{|x_1| \leq \epsilon\}\)

\[
\left| \int (e(x, x, 0) - \mathcal{E}_4^{MW}(x, 0) - \mathcal{E}_\text{corr}^{MW}(x, 0)) \psi(x) \, dx \right| \leq C \mu^{-1/2} h^{-3}. \tag{51}
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Here \(\mathcal{E}_\text{corr}^{MW} = O(\mu^{5/4} h^{-3/2})\) is associated with periodic zone \(\{|x_1| \leq c \mu^{-1/2}\}\), and is the sum of similar expressions in 2D case for \(V_\beta = V - (2\beta + 1) \mu hf_2\) with \(\beta \in \mathbb{Z}^+\); locally all of them but one could be dropped.
As $d = 4$ and magnetic field is non-degenerate and generic,

- remainder estimate is $O(\mu^{-1} h^{1-d})$ for generic $V$;
- remainder estimate is $O(\mu^{-1} h^{1-d} + \mu^2 h^{-2})$ for general $V$. 