Spectral Asymptotics for 2-dimensional Schrödinger Operator with Strong Degenerating Magnetic Field

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Preface

We consider spectral asymptotics of

$$A = \frac{1}{2}\Bigl(\sum_{j,k} P_j g^{jk}(x) P_k - V\Bigr), \qquad P_j = hD_j - \mu V_j \tag{1}$$

where g^{jk}, V_j, V are smooth real-valued functions of $x \in \mathbb{R}^2$ and (g^{jk}) is positive-definite matrix, $0 < h \ll 1$ is a Planck parameter and $\mu \gg 1$ is a coupling parameter. We assume that A is self-adjoint operator.

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2-dimensional magnetic Schrödinger is very different from 3-dimensional, all others could be close to one of this cases but are more complicated.

Important role is played by intensity of magnetic field

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We are interested in asymptotics of

$$\int e(x,x,0)\psi(x)\,dx \tag{3}$$

as $h \to +0$, $\mu \to +\infty$ where $e(x,y,\tau)$ is the Schwartz kernel of the spectral projector and $\psi(x)$ is cut-off function. Everything is assumed to be C^∞.

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$$\mathcal{E}^{\mathrm{MW}} = \frac{1}{2\pi}\sum_{n\geq 0} \theta\bigl(\tau - V - (2n+1)\mu h F\bigr)\mu h^{-1} F\sqrt{g} \qquad (4)$$

(with $\theta(\tau) = 0, 1$ as $\tau \leq 0$, $\tau > 0$ respectively) which implies that one should assume that $\mu h \leq C$.

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both are generic. Then $\Sigma = \{F = 0\}$ is C^∞ curve and by change of variables we can achieve locally $\Sigma = \{x_1 = 0\}$; then $F \asymp x_1$.

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Then by change of variables and the gauge transform I can achieve

$$A = \frac{1}{2}\beta\Big(h^2 D_1\alpha^2 D_1 + (hD_2 - \mu\phi(x)x_1^{\nu}/\nu)^2 - W\Big)\beta \tag{8}$$

with

$$\phi = \alpha = 1 \qquad \text{at } \Sigma \tag{9}$$

and $W = V/\beta^2$.

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(i) $1 \ll \mu \le h^{-1}$ when the answer is of magnitude h^{-2} and
(ii) $h^{-1} \le \mu \le Ch^{-\nu}$ when the answer is of magnitude $h^{-2}(\mu h)^{1/(\nu-1)}$ and it is provided by a strip

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One can prove that for $\mu \ge Ch^{-\nu}$ expression (3) is $O(\mu^{-s})$. What about remainder estimate in cases (i),(ii)?

Quick and dirty answer is provided by rescaling technique:
Consider ball $B(x, \gamma/2)$ with $\gamma = |x_1|$. Rescale it to $B(0, 1)$.

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Summation over partition in the outer zone

$$\mathcal{Z}_{\mathrm{out}} = \{\bar{\gamma} \le |x_1| \le \bar{\gamma}_1\} \tag{10}$$

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We want better!

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This is what my talk about

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Classical Dynamics

The insight to the problem and the way of the proof shows the classical dynamics. Assume for simplicity that $g^{jk} = \delta_{jk}$. Then

- If $F = \mathrm{const}$, $V = \mathrm{const}$ particles (on energy level 0) move clock-wise along circles of the radii $\sqrt{V}/\mu F$; the angular velocity is μF.
- If $F = \mathrm{const}$ and V is linear trajectories become hypercycloids because in addition to cyclotron movement appears the drift with the velocity $-(\nabla V)^{\perp}/2\mu F$ where $v^{\perp}$ means a rotated by angle $\pi/2$ counterclockwise.

- In the strong variable magnetic field cyclotron movement is supplemented by a drift along trajectories of

$$\frac{dx}{dt} = -\frac{1}{2\mu}\big(\nabla(V/F)\big)^{\perp} \tag{12}$$

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- In particular for model Hamiltonian

$$a_0 = \frac{1}{2}\Bigl(\xi_1^2 + (\xi_2 - \mu x_1^{\nu}/\nu)^2 - 1\Bigr) \tag{13}$$

the drift equation is

$$\frac{dx_1}{dt} = 0, \qquad \frac{dx_2}{dt} = \frac{1}{2}(\nu - 1)\mu^{-1}x_1^{-\nu} \tag{14}$$

and for $|x_1| \gg \bar{\gamma}$ gives a proper description of the picture.

For model Hamiltonian (13) one can consider also 1-dimensional movement along x_1 with potential

$$\mathcal{V}(x_1; k) = (k - x_1^\nu/\nu)^2 - 1, \qquad k = \xi_2; \tag{15}$$

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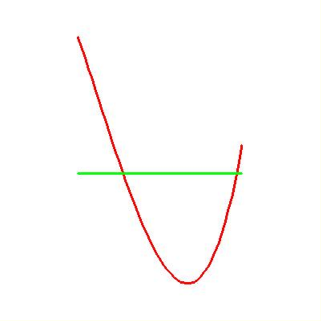
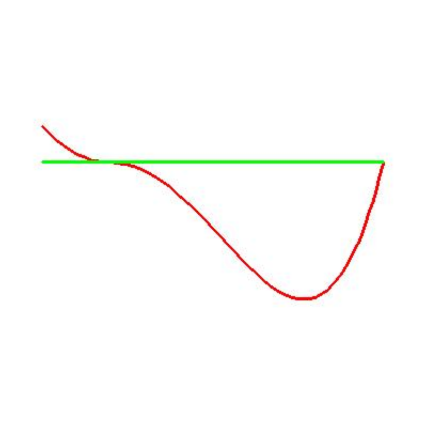
then for odd ν

- We have one-well potential:



- As $k = \pm 1$ one of its extremes is 0 where $\mathcal{V}' = 0$:



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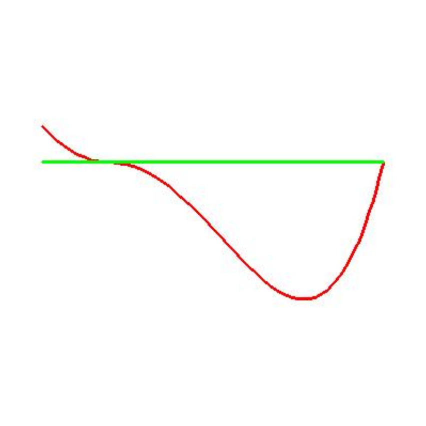
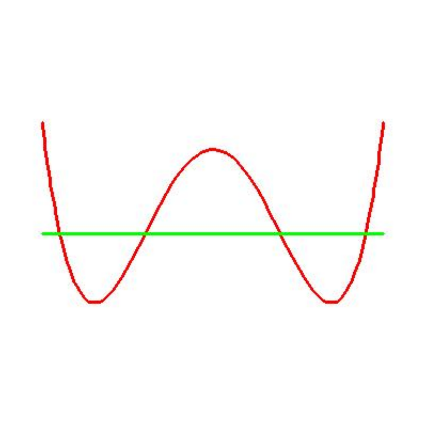


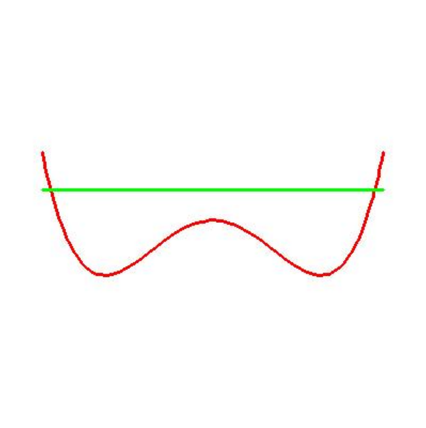
- As $k=0$ well becomes symmetric.

For even ν

- We have two-well potential if $k > 1$:



- But for $|k| < 1$ these wells join



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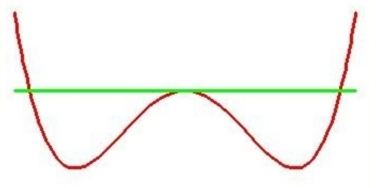

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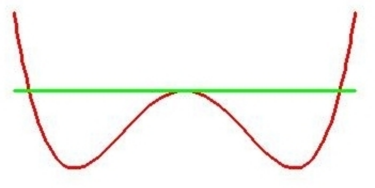

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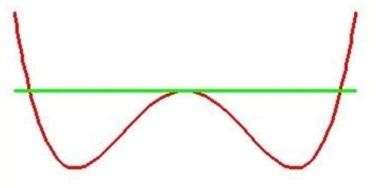

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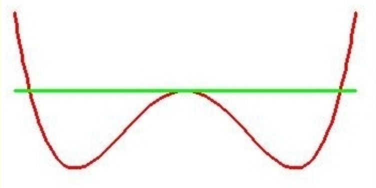

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Note that for $k \gg 1$ and even/odd ν the bottom of the wells/well is in $\pm(\nu k)^{1/\nu}$ / $(\nu k)^{1/\nu}$, depth is 1 and the width and area are $\asymp k^{-1+1/\nu}$.

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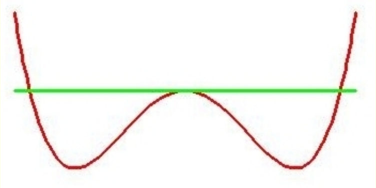

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Note that for $k\gg 1$ and even/odd ν the bottom of the wells/well is in $\pm(\nu k)^{1/\nu}$ / $(\nu k)^{1/\nu}$, depth is 1 and the width and area are $\asymp k^{-1+1/\nu}$. So, for $|k|\gg 1$ well is too narrow to keep a particle satisfying uncertainty principle.

So for $k \neq \pm 1$ the movement along x_1 is periodic with period

$$T(k) = 2 \int_{x_1^-(k)}^{x_1^+(k)} \frac{dx_1}{\sqrt{\mathcal{V}(x_1; k)}} \tag{16}$$

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but one needs to analyze the increment of x_2 during this period

$$I(k) = 2\int_{x_1^-(k)}^{x_1^+(k)} \frac{(k - x_1^\nu/\nu)dx_1}{\sqrt{\mathcal{V}(x_1;k)}}. \tag{17}$$

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There are trajectories on (x_1, x_2)-plane plotted by Maple for even ν:

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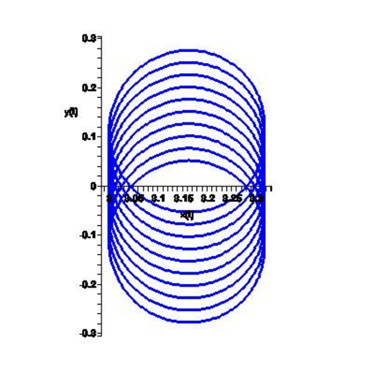


Figure: $k \gg 1$; as $x_1 > 0$ trajectory moves up and rotates clockwise

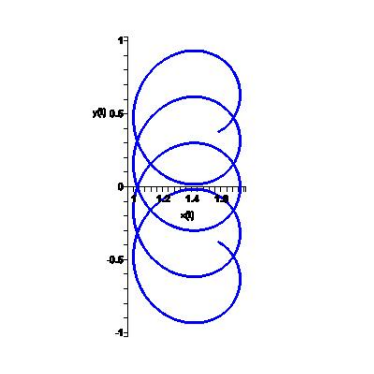


Figure: k decreases, still $k > 1$. Trajectory becomes less tight; actual size of cyclotrons increases and the drift is faster

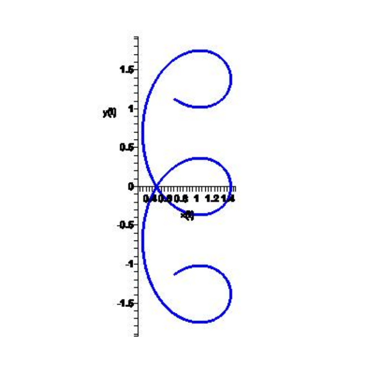


Figure: k further decreases, still $k > 1$. Trajectory becomes even less tight; actual size of cyclotrons increases and the drift is faster

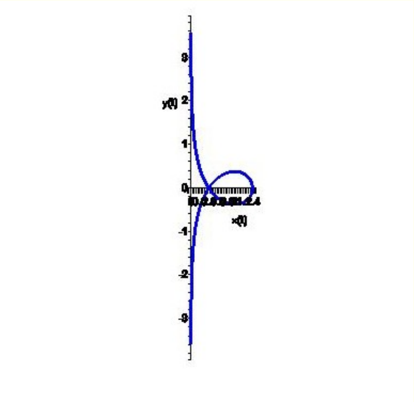
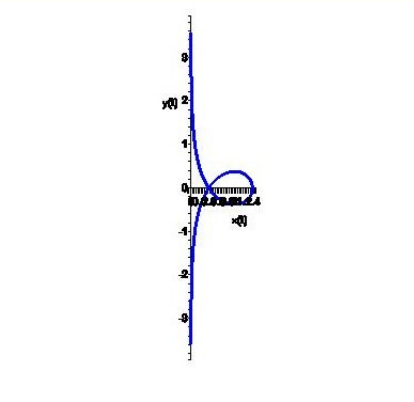


Figure: $k = 1$. Trajectory contains just one cyclotron



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These trajectories have mirror-symmetric as $x_1 < 0$ with movement up and counter-clockwise.

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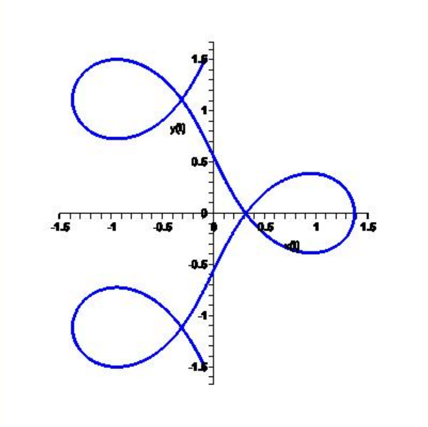


Figure: $k < 1$ slightly

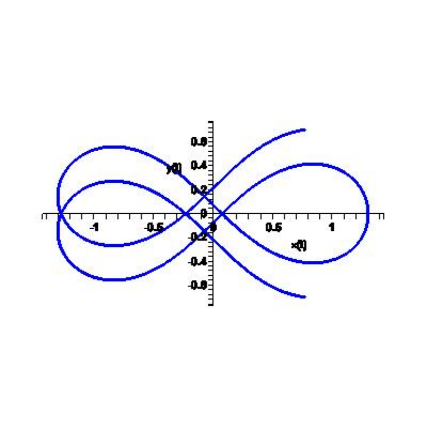


Figure: k further decays but still larger than k^*. Drift slows down

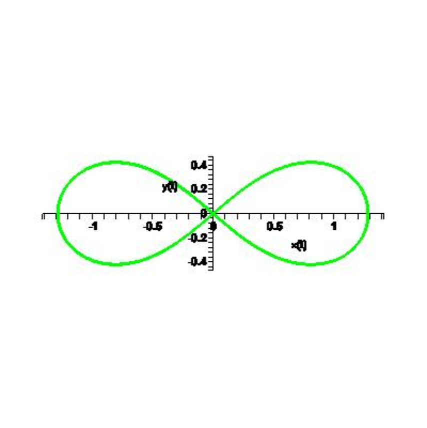


Figure: $k = k^*$. No drift; trajectory becomes periodic

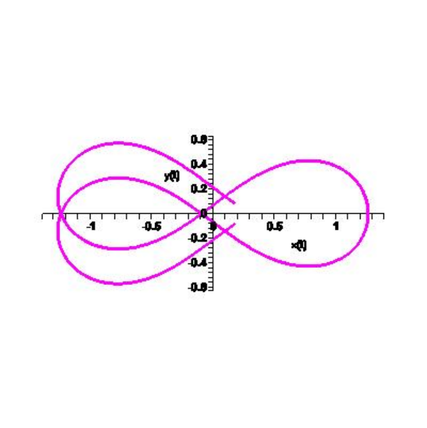


Figure: $k < k^*$. Drift now is down!

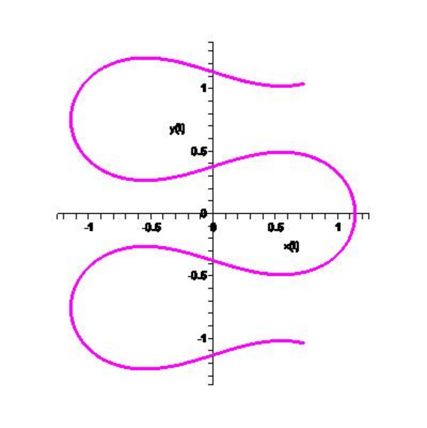
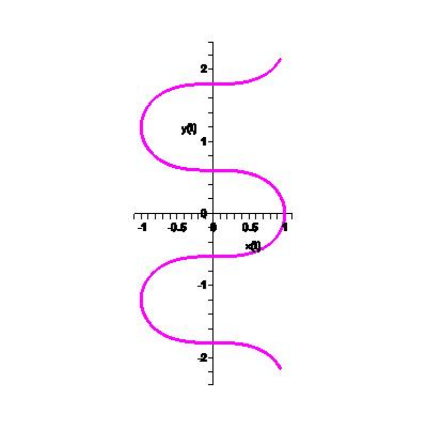


Figure: k decays further. Drift down accelerates.



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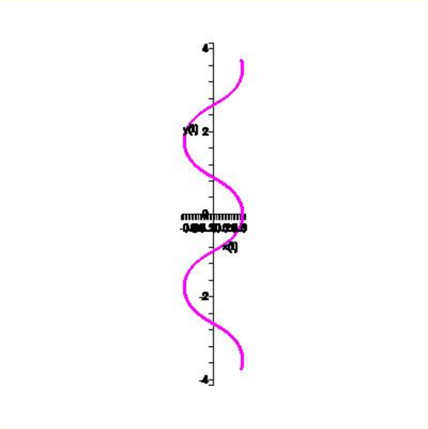

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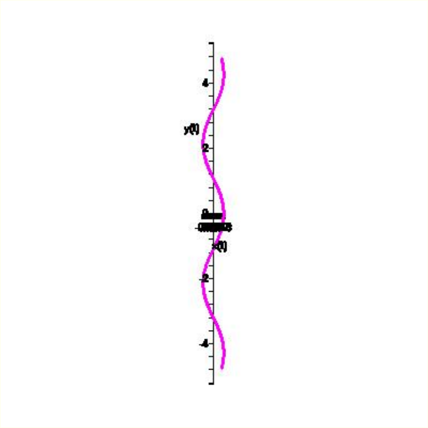

Figure: k almost -1 now

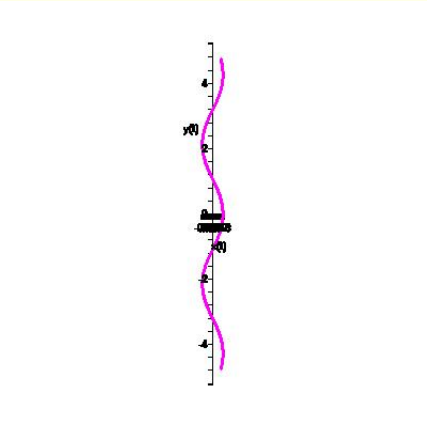

Figure: k almost -1 now

As $k = -1$ we have just straight line down

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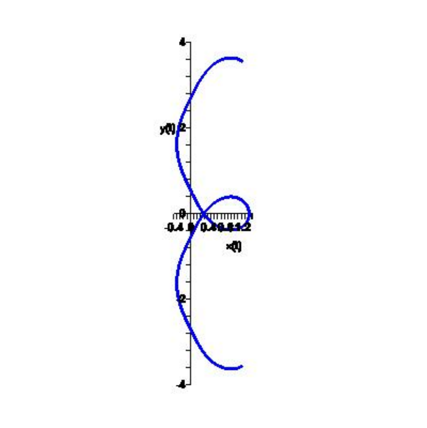

Figure: $k < 1$ slightly. Drift is up

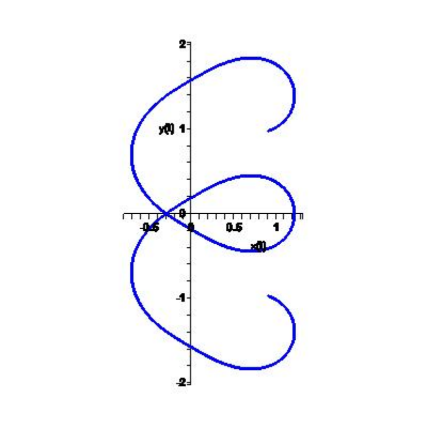

Figure: k decays. Drift up slows down

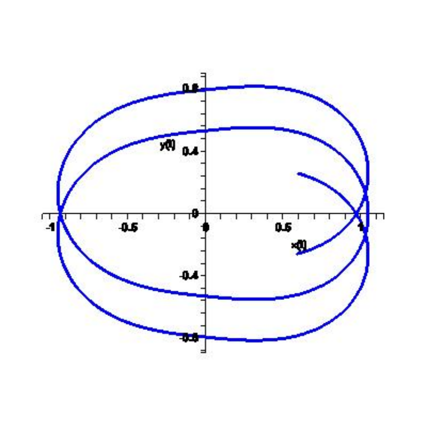

Figure: k decays further. Drift up slows down further

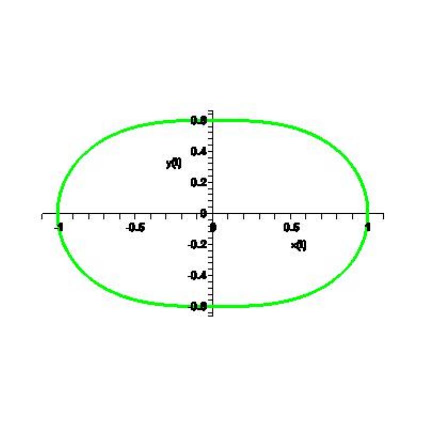

Figure: $k = 0$. No drift. Trajectory is periodic

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For the spectral asymptotics periodic trajectories are very important, especially short ones. Periodic trajectories shown above are very unstable and taking $V = 1 - \alpha x_1$ instead of x_1 sends breaks them down.

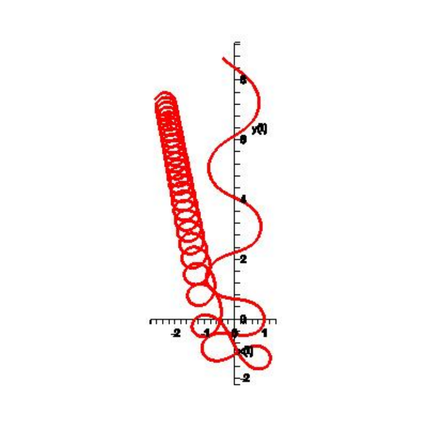

Figure: ν is even

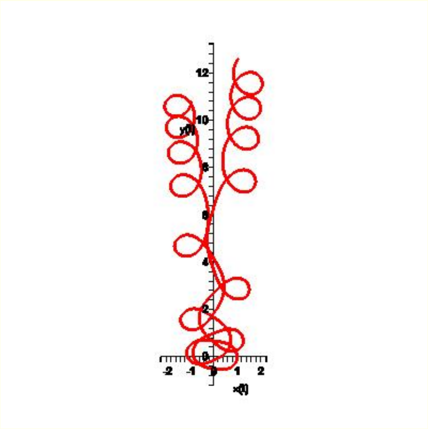

Figure: ν is odd

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From Classics to Semiclassics

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and remainder estimate $T^{-1}M$ with

$$M = \sup_{\tau:|\tau|\le\varepsilon} |F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma(Qu)| \tag{20}$$

where $u(x, y, t)$ is the Schwartz kernel of propagator $e^{ih^{-1}tA}$, where $u(x,y,t)$ is the Schwartz kernel of propagator $e^{ih^{-1}tA}$, $\bar{\chi} \in C_0^\infty(-1,1)$ equal 1 on $(-\frac{1}{2},\frac{1}{2})$, where $u(x, y, t)$ is the Schwartz kernel of propagator $e^{ih^{-1}tA}$, $\bar{\chi} \in C_0^\infty(-1, 1)$ equal 1 on $(-\frac{1}{2}, \frac{1}{2})$, $\chi_T(t) = \chi(t/T)$, where $u(x,y,t)$ is the Schwartz kernel of propagator $e^{ih^{-1}tA}$, $\bar{\chi} \in C_0^\infty(-1,1)$ equal 1 on $(-\frac{1}{2},\frac{1}{2})$, $\chi_T(t)=\chi(t/T)$, $\varepsilon = Ch|\log h|/T$.

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And there are two tricks:

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The first goal is achieved by Microlocal analysis which tells us that

$$|F_{t\to h^{-1}\tau}\bar{\chi}_{T'}(t-T)\Gamma(Qu)|\le Ch^s \tag{21}$$

with arbitrarily large exponent s provided

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Then $M \le Ch^{-d} \operatorname{vol}(\operatorname{supp} Q)\, T_0$ and we get a remainder estimate $Ch^{-d} \operatorname{vol}(\operatorname{supp} Q)\, T_0/T_1$.

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However one needs to remember that we are not in the classical but in the semiclassical situation when instead of individual trajectories we consider their beams with supp Q satisfying logarithmic uncertainty principle

$$\Delta x \cdot \Delta\xi \geq Ch|\log h|. \tag{22}$$

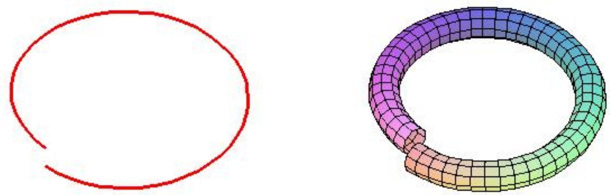

Figure: Classical and Semiclassical

We see that classical trajectory is not periodic but cannot say this about semiclassical beam until much larger time:

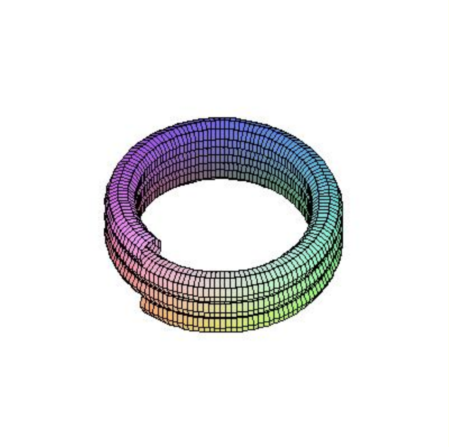

Figure: Semiclassics in long run

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Remainder estimate $O(h^{-1}\bar{\gamma}) = O(\mu^{-1/\nu}h^{-1})$ is our best shot.

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$$\rho \ge \bar{\rho}_1 = C\bigl(\bar{\gamma}^{-1} h|\log h|\bigr)^{\frac{1}{2}} \tag{24}$$

which leaves us with much smaller periodic zone

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Actually for the general operator rather than model one we need to assume that $\rho \ge C\bar{\gamma}$ but this does not spoil our dream estimate. So we need to consider periodic zone defined by (25) assuming that (26) does not hold.

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For operator (reminder of two early formulae)

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condition breaking periodicity will be

$$|\partial_{x_2} V| \asymp \zeta \ge C\max\bigl(\bar{\gamma}, h^{1-\delta}\bar{\gamma}^{-1}\bigr) \tag{27}$$

with arbitrarily small exponent $\delta > 0$

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So, under condition (27) remainder estimate will be $Ch^{-1}\bar{\gamma}$ again.
On the other hand, consider opposite case

$$|\partial_{x_2} V| \le \zeta \le C\bar{\gamma}^2. \tag{28}$$

Then for element Q supported in $\{|\xi_2 - k^* V^{1/2}| \asymp \rho\}$ (and in $B(0,1/2)$) with

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with any fixed m and estimate $O(h^{-1}\bar\gamma + h^{-\delta})$ with arbitrarily small exponent $\delta > 0$ without it.

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$$\frac{1}{2}\beta\Bigl(h^2 D_1 \alpha^2 D_1 + (\xi_2 - \mu\phi(x)x_1^\nu/\nu)^2 - W\Bigr)\beta \qquad (31)$$

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$$\frac{1}{2}\beta\Big(h^2 D_1 \alpha^2 D_1 + (\xi_2 - \mu\phi(x)x_1^\nu/\nu)^2 - W\Big)\beta \tag{31}$$

is nicely approximated by

$$\frac{1}{2}\Big(h^2 D_1^2 + (\xi_2 - \mu x_1^\nu/\nu)^2 - W(0, x_2)\Big) \tag{32}$$

step because x_1 is really small now and energy levels are close to 0.

After rescaling $x_1 \mapsto x_1/\bar{\gamma}$ (32) becomes $\mathbf{a}_0(\xi_2) - \frac{1}{2}W(0, x_2)$ with

$$\mathbf{a}_0(\xi_2) = \frac{1}{2}\Bigl(\hbar^2 D_1^2 + (\xi_2 - x_1^\nu/\nu)^2\Bigr), \qquad \hbar = h/\bar{\gamma}. \tag{33}$$

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As $\epsilon h^{-\nu} \le \mu \le C h^{-\nu}$ and $\hbar$ is not a small parameter anymore I do not know how eigenvalues $\lambda_n(\xi_2)$ of $\mathbf{a}_0(\xi_2)$ depend on ξ_2. In particular I don't know if stationary points are non-degenerate.

Otherwise I can recover $O(h^{-1}\bar{\gamma})$ remainder estimate under condition (30) with $m = 2$ i.e.

$$\sum_{1\le k\le 2} |\partial_{x_2}^k W| \ge \epsilon_1 \tag{34}$$

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and $O(h^{-\delta})$ otherwise.

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where we take different T_ι for different elements Q_ι of partition. So, we want to give more explicit formula but without losing remainder estimate realizing that

$$\int \mathcal{E}^{\mathrm{MW}}(x,0)\psi(x)\,dx \tag{36}$$

is the leading part of it.

In the outer zone (36) provides precision $O(\bar{\gamma}_1^{-1} + h^{-1}\bar{\gamma})$ with $\bar{\gamma}_1 = (\mu h)^{-1/(\nu-1)}$.

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$$(2\pi h)^{-1} \iiint \mathbf{e}(x_1, x_1, 0; x_2, \xi_2, \hbar)\psi(\frac{x_1}{\bar{\gamma}}, x_2)\, dx_1 dx_2 d\xi_2 \qquad (37)$$

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where $\mathbf{e}(x_1, y_1, \tau; x_2, \xi_2)$ is the Schwartz kernel of the spectral projector of 1-dimensional operator

$$\mathbf{a}(x_2, \xi_2, \hbar) = \frac{1}{2}\Bigl(\hbar^2 D_1 \alpha'^2 D_1 + \bigl(\xi_2 - \phi' x_1^\nu/\nu\bigr)^2 - W'\Bigr) \qquad (38)$$

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The simplest universally sharp formula is

$$\int \mathcal{E}^{\mathrm{MW}}(x, 0)\psi\, dx + \int \mathcal{E}^{\mathrm{MW}}_{\mathrm{corr}}(x_2, 0)\psi(0, x_2)\, dx_2 \qquad (40)$$

with

$$\mathcal{E}^{\mathrm{MW}}_{\mathrm{corr}}(x_2,\tau) = (2\pi h)^{-1}\int \mathbf{n}_0(\tau; x_2,\xi_2,\hbar)\,d\xi_2 - \int \mathcal{E}^{\mathrm{MW}}_0(\tau; x_1,x_2,\hbar)\,dx_1 \qquad (41)$$

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where $\mathbf{n}$ is an eigenvalue counting function for operator $\mathbf{a}_0$, $\mathbf{a}$ by replacing x_1 by 0 in α', ϕ', W':

$$\mathbf{a}_0(x_2,\xi_2,\hbar) = \frac{1}{2}\Bigl(\hbar^2 D_1^2 + \bigl(\xi_2 - x_1^\nu/\nu\bigr)^2 - \bar{W}(x_2)\Bigr) \qquad (42)$$

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Formula (40)-(41) is not completely explicit but is closer to it than even (39).

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$$\mathcal{E}^{\mathrm{MW}}_{\mathrm{corr}}(x_2,\tau) \approx \\ (2\pi h)^{-1} \int_{\{|\xi_2|\le C\}} \bigl(\mathbf{n}_0(\tau; x_2,\xi_2,\hbar) - n_0^{\mathrm{W}}(\tau; x_2,\xi_2,\hbar)\bigr)\, d\xi_2 \qquad (43)$$

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where n_0^{W} is Weyl approximation for $\mathbf{n}_0$.
Note $\mathbf{n}_0 - n_0^{\mathrm{W}} = O(1)$, but after integration with respect to ξ_2 we will gain $\hbar^{\frac{1}{2}}$ because of the character of degeneration of eigenvalue $\lambda_n \approx \bar{W}$ of $\mathbf{a}_0$ as $\xi_2 \approx k^*\bar{W}^{\frac{1}{2}}$.

So, there is sharp estimate

$$\mathcal{E}^{\mathrm{MW}}_{\mathrm{corr}}(x_2,\tau) = O(h^{-1}\hbar^{\frac{1}{2}}) = O(h^{-\frac{1}{2}}\bar{\gamma}^{-\frac{1}{2}}) \tag{44}$$

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but there is also integration with respect to x_2 which can reduce

$$\int \mathcal{E}^{\mathrm{MW}}_{\mathrm{corr}}(x_2,\tau)\bar{\psi}(x_2)\,dx_2 \tag{45}$$

to $O(h^{-1}\hbar^2 + \bar{\gamma}_1^{-1})$ in the non-degenerate case.

Using Bohr-Sommerfeld approximation one can calculate eigenvalues of $\mathbf{a}_0(\xi_2)$ with $O(\hbar^s)$ precision and $\mathcal{E}^{\mathrm{MW}}_{\mathrm{corr}}(x_2, \tau)$ with $O(h^{-1}\hbar^s)$ precision.

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In particular modulo $O(h^{-1}\hbar)=O(\bar\gamma^{-1})$

$$\mathcal{E}^{\mathrm{MW}}_{\mathrm{corr}}(x_2,0)\equiv \varkappa h^{-1}\hbar^{\frac{1}{2}}\bar W^{\frac{1}{4}-\frac{1}{4\nu}}G\Bigl(\frac{S_0\bar W^{\frac{1}{2}+\frac{1}{2\nu}}}{2\pi\hbar}\Bigr) \tag{46}$$

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with some constants $\varkappa$ and S_0 and function G defined by

$$G(t)=\int_{\mathbb{R}}\Bigl(t+\frac{1}{2}\eta^2-\lfloor t+\frac{1}{2}\eta^2+\frac{1}{2}\rfloor\Bigr)\,d\eta \tag{47}$$

with the converging integral in the right-hand expression.

One can prove easily that

$$G \not\equiv 0,\ G(t+1) = G(t),\ \int_0^1 G(t)\,dt = 0,\ G \in C^{\frac{1}{2}}. \tag{48}$$

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This is one of examples of short periodic trajectories really contributing to asymptotics.

This talk was prepared using beamer package for LaTeX. Figures were plotted by Maple.