CRM Workshop
on Spectral Geometry

Sharp Spectral Asymptotics for Magnetic
Schrödinger Operator

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1 Problem and Results

1.1 Preface

This talk presents much newer results. Important parts of the research were done in July 2002 when I was visiting Université Paris-Nord and in March 2003 during my stay at Banff International Research Station.

The results are summarized in

Sharp spectral asymptotics for operators with irregular coefficients. III. Schrödinger operator with a strong magnetic field. – Submitted to Comm. Part. Diff. Equats;
Sharp Spectral Asymptotics for Operators with Irregular Coefficients. IV. Multidimensional Schrödinger operator with a strong magnetic field.
Full-rank case. – Finished;

Sharp Spectral Asymptotics for Operators with Irregular Coefficients. V. Multidimensional Schrödinger operator with a strong magnetic field.
Not-Full-rank case. – In Process.

This talk is basically a survey of the results of these papers.
1.2 Operator

Either

\[
\sum_j P_j^2 + V(x), \quad P_j = \hbar D_j - \mu A_j(x)
\]  \hspace{1cm} (1)

with linear functions \( A_j(x) \) (the standard Schrödinger operator with the constant magnetic field), or more general
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with linear functions \( A_j(x) \) (the standard Schrödinger operator with the constant magnetic field), or more general

\[ \sum_{j,k} P_j g^{jk}(x) P_k + V(x), \quad P_j = \hbar D_j - \mu A_j(x) \] (2)

with general Riemannian metrics and vector potential \((A_1, \ldots, A_d)\).
In dimensions $d = 2, 3$ the second case is much more difficult technically, but in dimensions $d \geq 4$ results are very different too. I am talking only about (1).
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Another classification: rank $F = d$ and rank $F < d$ where $F = (F_{jk})$, $F_{jk} = \partial_k A_j - \partial_j A_k$, is a matrix intensity of magnetic field.
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Another classification: $\text{rank } F = d$ and $\text{rank } F < d$ where $F = (F_{jk})$, $F_{jk} = \partial_k A_j - \partial_j A_k$, is a matrix intensity of magnetic field.

If $d = 2$ then $F$ is reduced to a (pseudo)scalar, and only its absolute value is important. If $d = 3$ then $F$ is reduced to a (pseudo)vector, and only its length is important. But for $d \geq 4$ the whole structure of $F$ is important.
1.4 Special case $V = \text{const in } \mathbb{R}^d$

Then everything is explicit: in appropriate coordinates operator is

$$H = \sum_{1 \leq j \leq r} (h^2 D_j^2 + (hD_{j+r} - \mu f_j x_j)^2) + \sum_{1 \leq k \leq q} h^2 D_{2r+k}^2 \tag{3}$$

where $\text{rank } \mathcal{F}^r = 2r$ and $q = d - 2r$. 


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where $\text{rank } F^r = 2r$ and $q = d - 2r$. Then $h$-Fourier transform with respect to $(x_{r+1}, \ldots, x_d)$ transforms it into

$$\sum_{1 \leq j \leq r} (h^2 D_j^2 + (\xi_{j+1} - \mu f_j x_j)^2) + \sum_{1 \leq k \leq q} \xi_{2r+k}^2$$
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changing variables $x_j \mapsto x_j - \mu^{-1} \xi_{2j+1}$ ($j = 1, \ldots, r$) we get

$$\sum_{1 \leq j \leq r} \left( h^2 D_j^2 + \mu^2 f_j^2 x_j^2 \right) + \sum_{1 \leq k \leq q} \xi_{2r+k}^2$$
and then for $q = 0$ operator has pure point spectrum of infinite multiplicity and for $q \geq 1$ it has continuous spectrum.
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In all these cases

$$
e(x, x, \tau) = E_{d,r}^{MW}(\tau) =$$

$$
\omega_q(2\pi)^{-q} \sum_{\alpha \in \mathbb{Z}^+} \left( \tau - \sum_j (2\alpha_j + 1)f_j \mu h - V \right)^{\frac{q}{2}} \times
$$

$$
\mu^r h^{-d+r} f_1 \cdots f_r \quad (5)
$$

where $e(x, y, \tau)$ is the Schwartz’ kernel of the spectral projector and $\pm i f_1, \ldots, \pm i f_r$ are eigenvalues of $F$, $f_j > 0$. 
Note that the the number of terms is \( \asymp (\mu h)^{-r} \) and 
\[
E_{d,r}^{\text{MW}}(\tau) \asymp h^{-d} \quad \text{as} \quad \mu h \leq 1, \quad \tau - V \asymp 1 \quad \text{and the number of terms is} \quad \asymp 1 \quad \text{and} \quad E_{d,r}^{\text{MW}}(\tau) \asymp \mu^r h^{r-d} \quad \text{as} \quad \mu h \geq 1 \quad \text{and} \quad \tau - V - \sum_j f_j \mu h \asymp 1.
\]
Note that the number of terms is \( \asymp (\mu h)^{-r} \) and

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\]

\( \asymp 1 \) and \( E_{d,r}^{\text{MW}}(\tau) \asymp \mu^r h^{r-d} \) as \( \mu h \geq 1 \) and \( \tau - V - \sum_j f_j \mu h \asymp 1 \).

Also note that as \( \mu h \to +0 \) the \( r \)-dimensional Riemann sum becomes an integral and we get a standard Weyl expression:

\[
E_{d,r}^{\text{W}} = \omega_d (2\pi)^{-d} h^{-d} (\tau - V)^{\frac{d}{2}}. \quad (6)
\]
1.5 Classical Dynamics in Special Case

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- as $d = 3$ particles move along spirals and similar happens for rank $F < d$.

So, we see big difference between full- and not-full-rank cases and more subtle difference between $d = 2, 3$ and $d \geq 4$. 
1.6 Problem

Find asymptotics of $\int \psi(x)e(x, x, \tau) \, dx$ as $h \to +0$, $\mu \to +\infty$ where we assume that operator is self-adjoint and $\psi(x)$ is a cut-off function in the ball $B(0, 1)$. 
1.6 Problem

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Actually, $e(x, y, \tau)$ are Schwartz’ kernel of two framing approximations $\tilde{H}^\pm$ of our original operator $H$, but the difference between principal parts of asymptotics for $\tilde{H}^\pm$ does not exceed announced remainder estimate.
1.7 What has been known?

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As $\text{rank } F = d \geq 4$ M.Dimassi established sharp spectral asymptotics as $\mu \hbar \gg 1$.

In the general case G.Raikov got asymptotics but without any remainder estimate.
1.8 Standard results rescaled

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Changing variables $x \mapsto \mu x$ we arrive to standard situation with $h \mapsto \mu h$ and assuming that $\mu h \ll 1$ we get principal part $\asymp (\mu h)^{-d}\mu^d = h^{-d}$ and the remainder estimate $O((\mu h)^{1-d}\mu^d) = O(\mu h^{1-d})$. 
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These results are not sharp but the standard approach is very useful when magnetic field is not strong.
1.9 Classical Dynamics (again)

Since for $\mu h \ll 1$ classical dynamics describes well quantum dynamics i.e. dynamics of $e^{ih^{-1}tH}$, classical dynamics is important to understand spectral asymptotics.
Full-Rank case

As \( d = 2 \) trajectories are perturbed small circles (of radii \( \approx \mu^{-1} \)) drifting with the speed \( \leq C\mu^{-1} \) along level lines of \( V \) (so, in direction orthogonal to electric field):
**Full-Rank case**

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Here electric field is constant:

and similar picture holds for variable electric field.
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![Diagram](image)

and similar picture holds for variable electric field.

As \( d = 2r \) trajectories are perturbed small orbits (described above; of radii \( \approx \mu^{-1} \)) drifting with the speed \( \leq C\mu^{-1} \) along trajectories of \( \frac{d}{dt} x_j = \sum_k \Phi^{jk} \partial_k V \) where \( (\Phi^{jk}) = (F_{jk}^{-1}) \).
So, we can follow these trajectories until time $T_1 = \epsilon \mu$ and then we hope that the Tauberian methods will give us remainder estimate $O(\mu^{-1} h^{1-d})$. 
Basically this is a correct guess (for sufficiently smooth $V$ and $\mu h \leq 1$) but we will need nondegeneracy condition

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The simplest explanation is that in the special case $\text{rank } F = d$, $\mathbb{R}^d$, $V = \text{const}$, $\int E^{MW} \psi(x)(x, \tau) \, dx$ jumps by $\simeq \mu h^{1-d}$ when $\tau$ passes through Landau levels $V + \sum (2\alpha_j + 1)f_j\mu h$ and this is the worst case scenario!
Not-Full-Rank case

In this case major movement is along free variables $x_{\text{free}} = (x_{2r+1}, \ldots, x_d)$ and its speed is of magnitude $|\xi_{\text{free}}|$. 

- Normal Helix along $z$
- Electric Field along $y$
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Again, basically this is a correct guess but we need some smoothness and to some degree we need (7).
Super-Strong Magnetic Field

Case $\mu h \geq 1$ is different and we discuss it later. However, the answer is essentially the same: while the principal part of asymptotics is $\asymp \mu^r h^{-d+r}$, the remainder estimate is (under proper conditions) $O(\mu^{r-1} h^{1-d+r})$ in full-rank case and $O(\mu^r h^{1-d+r})$ in non-full-rank case,
Super-Strong Magnetic Field

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2 Weak Magnetic Field

2.1 Full-Rank case

So, let us consider operator (1) and make $\varepsilon$-approximation. Error in $V$ is $O(\varepsilon^l |\log \varepsilon|^\sigma)$ provided $V \in C^{l,\sigma}$
2 Weak Magnetic Field

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So, let us consider operator (1) and make $\varepsilon$-approximation. Error in $V$ is $O(\varepsilon^l \log \varepsilon |\sigma|$) provided $V \in C^{l,\sigma}$. \(^a\)

\(^a\) $V \in C^{l,\sigma}$ means that the derivatives of order $\lfloor l \rfloor$ of $V$ are continuous for continuity modulus $\varpi(t) = t^{l-\lfloor l \rfloor} |\log t|^{-\sigma}$ (unless $l \in \mathbb{Z}$, $\sigma < 0$ when definition is obviously modified).
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So, let us consider operator (1) and make $\varepsilon$-approximation. Error in $V$ is $O(\varepsilon^l | \log \varepsilon | ^\sigma)$ provided $V \in C^{l,\sigma}$.

Approximation error in Weyl expression is of magnitude $O(\varepsilon^l | \log \varepsilon | ^\sigma h^{-d})$.

\footnote{$V \in C^{l,\sigma}$ means that the derivatives of order $[l]$ of $V$ are continuous for continuity modulus $\omega(t) = t^{l-[l]} | \log t | ^{-\sigma}$ (unless $l \in \mathbb{Z}$, $\sigma < 0$ when definition is obviously modified).}
Now, consider evolution of $e^{i\hbar^{-1}tH}$ until time $T_0$. To do it one should note first that

$$[P_j, P_k] = i\mu\hbar F_{jk}, \quad [P_j, x_k] = -i\hbar \delta_{jk}$$ \hspace{1cm} (8)

and introduce new variables $X_j = x_j - \mu^{-1} \sum \Phi^{jk} P_k$ s.t.

$$[P_j, X_k] = 0, \quad [X_j, X_k] = i\mu^{-1} \hbar \Phi^{jk}.$$ \hspace{1cm} (9)
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They evolve with the speed $O(\mu^{-1})$ in the classical propagation and for time $T$ drift is $\asymp \mu^{-1} T$ under condition (7).
This statement could be done and justified in quantum sense too, provided we satisfy logarithmic uncertainty principle:

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\]

This would mean that trace of \( \phi(hD_t)e^{ih^{-1}tH} \) will be negligible for \( T \leq |t| \leq T_1 = \epsilon \mu \) (\( \phi \) is supported in small vicinity of 0; we make cut-off on energies). It happens because singularities leave the diagonal. I remind that \( T_1 \) is a time until which we can follow evolution.
Remarks. (i) Why two scales in $X$? – Because of (9) these variables are self-dual; also $\mu^{-1} \hbar$ is semiclassical constant now.
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Remarks. (i) Why two scales in $X$? – Because of (9) these variables are self-dual; also $\mu^{-1}h$ is semiclassical constant now.

(ii) Why not $\varepsilon \times \varepsilon \geq C\mu^{-1}h|\log h|$? – Because we manage to avoid this in our proof based on energy estimates.

(iii) Why not $\mu^{-1}T^{-1}T \geq C\mu^{-1}h|\log h|$? – Because we need a shift just in one variable.
Now back to standard results rescaled. Consider evolution rather than asymptotics. In standard situation we could follow evolution until time $T_{\text{standard}0} = \epsilon$. 
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So, until time $T_0$ standard theory is applicable.
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So, until time $T_0$ standard theory is applicable. To connect standards results rescaled and arguments above $T_0$ should satisfy (10):

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So, until time $T_0$ standard theory is applicable. To connect standards results rescaled and arguments above $T_0$ should satisfy (10):

$$\epsilon \times \mu^{-1} T_0 \geq C \mu^{-1} h | \log h|$$

(10)

or

$$\epsilon = C \mu h | \log h|.$$  \hspace{1cm} \text{(11)}

We took the smallest $\epsilon$. 
Then everything going well and we get asymptotics with remainder estimate

\[ C \mu^{-1} h^{1-d} + C(\mu h | \log h|)^l | \log h|^{-\sigma} h^{-d} \]
Then everything going well and we get asymptotics with remainder estimate

$$C \mu^{-1} h^{1-d} + C (\mu h | \log h |)^l | \log h |^{-\sigma} h^{-d}$$

where the first term is a correct remainder estimate.
Then everything going well and we get asymptotics with remainder estimate

\[ C \mu^{-1} h^{1-d} + C(\mu h | \log h |)^l | \log h |^{-\sigma} h^{-d} \]

where the first term is a correct remainder estimate and the second term is an approximation error.
The principal part of this asymptotics is given by universal but rather implicit formula

\[-h^{-1} \int_{-\infty}^{\tau} F_{t \to h^{-1} \tau'} \bar{\chi}(\frac{t}{T}) \text{Tr}(e^{ih^{-1}tH} \psi) \, d\tau' \quad (12)\]

where \( \bar{\chi} \) is supported in \((-1, 1)\) and equal 1 in \((-\frac{1}{2}, \frac{1}{2})\) and \(T = T_0\) (or larger)
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and after some calculations we can replace it by magnetic Weyl expression \( \int E^{MW}(\tau, x) \psi(x) \, dx \) as \( \mu \leq h^{\delta-1} \)
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and after some calculations we can replace it by magnetic Weyl expression \(\int E^{\text{MW}}(\tau, x) \psi(x) \, dx\) as \(\mu \leq h^{\delta-1}\)

and by standard Weyl expression \(\int E^{\text{W}}(\tau, x) \psi(x) \, dx\) as \(\mu \leq h^{-\frac{1}{2}}\)
Thus:

**Theorem 1** Let $d = 2r$ and condition (7) hold. Then for

$$\mu \leq h^{\delta-1}$$  \hspace{1cm} (13)

with an arbitrarily small exponent $\delta > 0$ the following estimate holds

$$\left| \int (e(x, x, \tau) - E^{\text{MW}}(x, \tau)) \psi(x) \, dx \right| \leq C \mu^{-1} h^{1-d} + C(\mu h | \log h|)^l | \log h|^{-\sigma} h^{-d}$$  \hspace{1cm} (14)
Thus:

**Theorem 1** Let \( d = 2r \) and condition (7) hold. Then for \( \mu \leq h^{\delta-1} \) (13) with an arbitrarily small exponent \( \delta > 0 \) the following estimate holds

\[
| \int (e(x, x, \tau) - E^{MW}(x, \tau))\psi(x) \, dx | \leq C \mu^{-1} h^{1-d} + C(\mu h |\log h|)^l |\log h|^{-\sigma} h^{-d} \]  

(14)

Look at the second term in the remainder estimate (14). If \( l \) is large, it will be small. So, in the smooth case we got sharp remainder estimate if (7) holds.
What about non-smooth case? In the next section we consider a different approach with $\varepsilon = C(\mu^{-1} h|\log h|)^{\frac{1}{2}}$ as $d = 2$ and $\varepsilon = C\mu^{-1}$ as $d = 2, 4, \ldots$. 
What about non-smooth case? In the next section we consider a different approach with $\varepsilon = C(\mu^{-1}h|\log h|)^{\frac{1}{2}}$ as $d = 2$ and $\varepsilon = C\mu^{-1}$ as $d = 2, 4, \ldots$.

Present choice of $\varepsilon = C\mu h|\log h|$ is better iff

$$\mu \leq h^{-\frac{1}{3}}|\log h|^{\frac{1}{3}} \text{ for } d = 2,$$
$$\mu \leq h^{-\frac{1}{2}}|\log h|^{\frac{1}{2}} \text{ for } d = 4, 6, \ldots,$$
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and then we formulate

**Corollary 2** *In frames of theorem 1 sharp remainder estimate $O(\mu^{-1}h^{1-d})$ holds for $\mu \leq h^{-\frac{1}{3}}|\log h|^{\frac{1}{3}}$ as $V \in C^{2,1}$ and for $\mu \leq h^{-\frac{1}{2}}|\log h|^{\frac{1}{2}}$ as $V \in C^{3,\frac{3}{2}}$.***
2.2 Not-Full-Rank case

Now we have free variables $x_{\text{free}} = (x_{2r+1}, \ldots, x_d)$ and we do not need condition (7) at this stage. Consider evolution along free variables.
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Now we have free variables \( x_{\text{free}} = (x_{2r+1}, \ldots, x_d) \) and we do not need condition (7) at this stage. Consider evolution along free variables. The speed is \( \sim |\xi_{\text{free}}| \) and the shift for time \( T \) is \( \sim |\xi_{\text{free}}|T \) and to make it observable we need to satisfy logarithmic uncertainty principle

\[
\min\left( |\xi_{\text{free}}|T, \varepsilon \right) \times |\xi_{\text{free}}| \geq C h |\log h| \tag{15}
\]
2.2 Not-Full-Rank case

Now we have free variables \( x_{\text{free}} = (x_{2r+1}, \ldots, x_d) \) and we do not need condition (7) at this stage. Consider evolution along free variables. The speed is \( \simeq |\xi_{\text{free}}| \) and the shift for time \( T \) is \( \simeq |\xi_{\text{free}}|T \) and to make it observable we need to satisfy logarithmic uncertainty principle

\[
\min\left( |\xi_{\text{free}}|T, \varepsilon \right) \times |\xi_{\text{free}}| \geq C h |\log h| \tag{15}
\]

and this means that we should take

\[
T = C h |\log h| \times |\xi_{\text{free}}|^{-2}, \quad \varepsilon = C h |\log h| \times |\xi_{\text{free}}|^{-1}. \tag{16}
\]
Since we want to plug $T_0 = \epsilon\mu^{-1}$ (from standard theory rescaled, as before) we need to assume

$$|\xi_{\text{free}}| \geq \bar{\varrho} = (C\mu h|\log h|)^{\frac{1}{2}} + C\mu^{-1};$$

(17)
Since we want to plug $T_0 = \epsilon \mu^{-1}$ (from standard theory rescaled, as before) we need to assume

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we added $C \mu^{-1}$ because for $|\xi_{\text{free}}| \leq C \mu^{-1}$ evolution with respect to other variables could be faster.
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we added $C \mu^{-1}$ because for $|\xi_{\text{free}}| \leq C \mu^{-1}$ evolution with respect to other variables could be faster. Then we need to plug $T_1 = \epsilon \varrho, \varrho = |\xi_{\text{free}}|$ because for larger $|t|$ singularity can turn back. Actually we can take $T_1 = \epsilon \varrho |\log \varrho|^2$ as $(l, \sigma) \succeq (1, 2)$ because we can choose time direction arbitrarily.
This leads to contribution of outer zone $\{|\xi_{\text{free}}| \geq \bar{\varrho}\}$ to the remainder estimate

$$Ch^{1-d} \int |\log \varrho|^{-\sigma} d\varrho^q \leq Ch^{1-d}$$
This leads to contribution of outer zone \( \{ |\xi_{\text{free}}| \geq \tilde{\varrho} \} \) to the remainder estimate

\[
Ch^{1-d} \int |\log \varrho|^{-\sigma} \, d\varrho^q \leq Ch^{1-d}
\]

and to contribution this zone to the approximation error

\[
Ch^{-d} \int (\frac{h}{\varrho})^l |\log \frac{h}{\varrho}|^{-\sigma} \, d\varrho^q \leq Ch^{1-d}
\]

as \((l, \sigma) = (1, 2)\) for \(q = 1\) and \((l, \sigma) = (1, 1)\) for \(q \geq 2\); \(\varrho^q\) appears as a volume of the layer \( \{ \varrho \leq |\xi_{\text{free}}| \leq 2\varrho \} \).
This leads to contribution of outer zone \( \{ |\xi_{\text{free}}| \geq \varrho \} \) to the remainder estimate

\[
Ch^{1-d} \int |\log \varrho|^{-\sigma} \, d\varrho^q \leq Ch^{1-d}
\]

and to contribution this zone to the approximation error

\[
Ch^{-d} \int \left( \frac{h}{\varrho} \right)^l |\log \frac{h}{\varrho}|^{-\sigma} \, d\varrho^q \leq Ch^{1-d}
\]

as \((l, \sigma) = (1, 2)\) for \(q = 1\) and \((l, \sigma) = (1, 1)\) for \(q \geq 2\); \(\varrho^q\) appears as a volume of the layer \(\{ \varrho \leq |\xi_{\text{free}}| \leq 2\varrho \}\).

Note, that we took \(\varepsilon\) depending on \(\xi\) here. It is easy to do everything rigorously.
Consider now inner zone \( \{ |\xi_{\text{free}}| \leq \bar{\varrho} \} \).
Consider now inner zone \( \{ |\xi_{\text{free}}| \leq \bar{\varrho} \} \).

One can take \( T_1 = \epsilon \mu^{-1} \) and \( \epsilon = C \mu h |\log h| \) here from the standard theory rescaled.
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One can take $T_1 = \epsilon \mu^{-1}$ and $\epsilon = C \mu h |\log h|$ here from the standard theory rescaled.

Then the contribution of this zone to the remainder estimate is $C \mu h^{1-d} \bar{\varrho}^q$; its contribution to the approximation error is smaller.
Consider now inner zone \(|\xi_{\text{free}}| \leq \bar{\varrho}\).

One can take \(T_1 = \epsilon \mu^{-1}\) and \(\epsilon = C \mu h|\log h|\) here from the standard theory rescaled.

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As \(q = 1\) this is better than \(C h^{1-d}\) iff \(\mu \leq h^{-\frac{1}{3}}|\log h|^{-\frac{1}{3}}\).
Consider now inner zone \( \{ |\xi_{\text{free}}| \leq \bar{\varrho} \} \).

One can take \( T_1 = \epsilon \mu^{-1} \) and \( \epsilon = C \mu h|\log h| \) here from the standard theory rescaled.

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As \( q = 1 \) this is better than \( Ch^{1-d} \) iff \( \mu \leq h^{-\frac{1}{3}}|\log h|^{-\frac{1}{3}} \); as \( q = 2 \) this is better than \( Ch^{1-d} \) iff \( \mu \leq h^{-\frac{1}{2}}|\log h|^{-\frac{1}{2}} \) etc.
Consider now inner zone \( \{ |\xi_{\text{free}}| \leq \varrho \} \).

One can take \( T_1 = \epsilon \mu^{-1} \) and \( \epsilon = C \mu h |\log h| \) here from the standard theory rescaled.

Then the contribution of this zone to the remainder estimate is \( C \mu h^{1-d} \varrho^q \); its contribution to the approximation error is smaller.

As \( q = 1 \) this is better than \( C h^{1-d} \) iff \( \mu \leq h^{-\frac{1}{3}} |\log h|^{-\frac{1}{3}} \); as \( q = 2 \) this is better than \( C h^{1-d} \) iff \( \mu \leq h^{-\frac{1}{2}} |\log h|^{-\frac{1}{2}} \) etc.

Actually one can improve this a bit:
Theorem 3  Let either $q = 1$, $V \in C^{1,2}$ or $q \geq 2$, $V \in C^{1,1}$. Then the following estimate holds

$$\left| \int (e(x, x, \tau) - E^{MW}(x, \tau)) \psi(x) \, dx \right| \leq C h^{1-d} + C (\mu h)^{\frac{q}{2} + 1} h^{-d}. \quad (18)$$

In particular, for $q = 1$, $\mu \leq h^{-\frac{1}{3}}$ and $q \geq 2$, $\mu \leq h^{-\frac{1}{2}}$ sharp remainder estimate $O(h^{1-d})$ holds.

Note, that at this stage extra smoothness is not very useful.
On the other hand, if for $\varrho \leq (\mu h | \log h)^{1/2}$ we take $T = h\varepsilon^{-1} | \log h |$, we will be able to use condition (7):

**Theorem 4** Let either $q = 1$, $V \in C^{l,\sigma}$ or $q \geq 2$, $V \in C^{1,1}$. Let non-degeneracy condition (7) be fulfilled. Then the following estimate holds

$$\left| \int (e(x, x, \tau) - E^{MW}(x, \tau)) \psi(x) \, dx \right| \leq C h^{1-d} + C(\mu h)^{\frac{q}{2}+l} | \log h |^{l-\sigma} h^{-d}. \quad (19)$$

In particular, for $q = 1$, $l = 3/2$, $\sigma = 1/2$, $\mu \leq h^{-\frac{1}{2}} | \log h |^{-1/2}$ sharp remainder estimate $O(h^{1-d})$ holds.
3 Stronger Magnetic Field: Reduction

3.1 Preliminaries

Now we need to reduce operator to canonical form. Smooth operators as $d = 2$ are reduced to

$$
\sum_{m+k \geq 1} \mu^{2-2k-2m} a_{m,k}(x_2, \mu^{-1} hD_2)\left(h^2 D_1^2 + \mu^2 x_1^2\right)^{m} 
$$

(20)
3 Stronger Magnetic Field: Reduction

3.1 Preliminaries

Now we need to reduce operator to canonical form. Smooth operators as $d = 2$ are reduced to

$$
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$$

and as $d = 3$ to

$$
\sum_{m+k+j \geq 1} \mu^{2-2k-2m-2j} a_{m,k,j}(x_2, x_3, \mu^{-1} hD_2) \times
\left( h^2 D_1^2 + \mu^2 x_1^2 \right)^m (hD_3)^{2j}.
$$
Now there are two obstacles to do this: non-smoothness of operator and resonances. Both of these obstacles are extremely nasty for operator (2).
Now there are two obstacles to do this: non-smoothness of operator and resonances. Both of these obstacles are extremely nasty for operator (2).

Since we first $\varepsilon$-mollify operator, symbols $a_*(x_2, \xi_2)$ etc are smooth in $\varepsilon$-scale only with respect to both variables and to have them as proper $\mu^{-1}h$-pdos one needs to assume the logarithmic uncertainty principle $\varepsilon^2 \geq C\mu^{-1}h|\log h|$: 

$$\varepsilon \geq C(\mu^{-1}h|\log \mu|)^{\frac{1}{2}}.$$  (22)
3.2 Main part

Let us consider reduction. Assuming that kinetic part $H_0$ of operator $H$ is of the form (3) we make Fourier transform $x'' \rightarrow \mu^{-1} h \xi''$ with respect to $x''$, where $x = (x'; x''; x''') = (x_1, \ldots, x_r; x_{r+1}, \ldots, x_{2r}; x_{2r+1}, \ldots, x_d)$. 
3.2 Main part

Let us consider reduction. Assuming that kinetic part $H_0$ of operator $H$ is of the form (3) we make Fourier transform $x'' \mapsto \mu^{-1} h \xi''$ with respect to $x''$, where

$x = (x'; x''; x''') = (x_1, \ldots, x_r; x_{r+1}, \ldots, x_{2r}; x_{2r+1}, \ldots, x_d)$.

Then $H$ is transformed into

$$
\sum_{1 \leq j \leq r} \left( h^2 D_j^2 + \mu^2 (\xi_{j+r} - f_j x_j)^2 \right) + \sum_{1 \leq k \leq q} h^2 D_{2r+k}^2 + V(x', \mu^{-1} h D \xi'', x''').
$$

(23)
Changing coordinates \( x'_{\text{new}} = \mu (\Phi^{-\frac{1}{2}} x' - \Phi^{\frac{1}{2}} \xi'') \) with

\[
\phi = \text{diag}(f_1^{-1}, \ldots, f_r^{-1})
\]

we get

\[
\sum_{1 \leq j \leq r} f_j (\mu^2 h^2 D_j^2 + x_j^2) + \sum_{1 \leq k \leq q} h^2 D_{2r+k}^2 + V(\mu^{-1} \Phi^{\frac{1}{2}} x' + \Phi x'', \mu^{-1} h D'', x''')
\]

where we redenoted \( \xi'' \) as \( x'' \).
Changing coordinates $x'_{\text{new}} = \mu(\Phi^{-\frac{1}{2}} x' - \Phi^{\frac{1}{2}} \xi'')$ with 

$\phi = \text{diag}(f_1^{-1}, \ldots, f_r^{-1})$ we get

$$
\sum_{1 \leq j \leq r} f_j \left( \mu^2 h^2 D_j^2 + x_j^2 \right) + \sum_{1 \leq k \leq q} h^2 D_{2r+k}^2 + V(\mu^{-1} \Phi^{\frac{1}{2}} x' + \Phi x'', \mu^{-1} h D'', x''') \quad (24)
$$

where we redenoted $\xi''$ as $x''$.

On energy levels below $\tau \leq c$ we have $|x'| \leq C_0$ and the main part of (24) is given by

$$
\sum_{1 \leq j \leq r} f_j \left( \mu^2 h^2 D_j^2 + x_j^2 \right) + \sum_{1 \leq k \leq q} h^2 D_{2r+k}^2 + V(\Phi x'', \mu^{-1} h D'', x''') \quad (25)
$$
Studying operator (25) is easy: decomposing in Hermite functions of $(\mu h)^{-\frac{1}{2}} x'$ we arrive to a family of operators

$$H_\alpha = \sum_{1 \leq k \leq q} h^2 D_{2r+k}^2 + V_\alpha(\Phi x'', \mu^{-1} h D'', x''')$$  \hspace{1cm} (26)$$

where

$$V_\alpha = V + \sum_{1 \leq j \leq r} (2\alpha_j + 1) \mu h f_j, \quad \alpha \in \mathbb{Z}^+ r.$$  \hspace{1cm} (27)$$
• If $q = 0$ (26) is a family of $\mu^{-1}h$-pdos.
• If \( q = 0 \) \((26)\) is a family of \( \mu^{-1}h\)-pdos. Without non-degeneracy condition \((7)\) there is no hope to get a remainder estimate better than \( O(\mu h^{1-d}) \): if for some \( \alpha \) \( V_\alpha \equiv \tau \) both principal part and remainder estimate for \( H_\alpha \) are of magnitude \( \mu^r h^{-r} \) (and the family contains \( \asymp (\mu h)^{-r} \) operators: for \( |\alpha| \geq C(\mu h)^{-1} \) energy levels of \( H_\alpha \) are too high to contribute).
• If $q = 0$ (26) is a family of $\mu^{-1} h$-pdos. Without non-degeneracy condition (7) there is no hope to get a remainder estimate better than $O(\mu h^{1-d})$: if for some $\alpha V_\alpha \equiv \tau$ both principal part and remainder estimate for $H_\alpha$ are of magnitude $\mu^r h^{-r}$ (and the family contains $\simeq (\mu h)^{-r}$ operators: for $|\alpha| \geq C(\mu h)^{-1}$ energy levels of $H_\alpha$ are too high to contribute).

However, under non-degeneracy condition (7) remainder estimate for $H_\alpha$ is $O(\mu^{r-1} h^{1-r})$ and the total remainder estimate is $O(\mu^{r-1} h^{1-r} \times (\mu h)^{-r}) = O(\mu^{-1} h^{1-d})$. 
• If $q \geq 1$ (26) is a family of Schrödinger operators with respect to free variables $x'''$ with potential which are $\mu^{-1}h$-pdos with respect to $x''$. 
• If $q \geq 1$ (26) is a family of Schrödinger operators with respect to free variables $x'''$ with potential which are $\mu^{-1}h$-pdos with respect to $x''$. For such operators remainder estimate cannot be too bad, especially for larger $q$, $l$, 
If \( q \geq 1 \) (26) is a family of Schrödinger operators with respect to free variables \( x''' \) with potential which are \( \mu^{-1}h \)-p dos with respect to \( x'' \). For such operators remainder estimate cannot be too bad, especially for larger \( q, l \), but it also cannot be too good: in the best case scenario it is \( O(h^{1-q} \times \mu^r h^r) \) while the principal part of asymptotics is of magnitude \( O(h^{-q} \times \mu^r h^r) \).
If $q \geq 1$ (26) is a family of Schrödinger operators with respect to free variables $x^{'''}$ with potential which are $\mu^{-1}h$-pdos with respect to $x''$. For such operators remainder estimate cannot be too bad, especially for larger $q, l$, but it also cannot be too good: in the best case scenario it is $O(h^{1-q} \times \mu^r h^r)$ while the principal part of asymptotics is of magnitude $O(h^{-q} \times \mu^r h^r)$. Multiplying by the number of operators we get remainder estimate $O(h^{1-d})$ (in the best case) and the principal part $\sim h^{-d}$. 
• If \( q \geq 1 \) (26) is a family of Schrödinger operators with respect to free variables \( x^{'''} \) with potential which are \( \mu^{-1} h \)-pdos with respect to \( x^{''} \). For such operators remainder estimate cannot be too bad, especially for larger \( q, l \), but it also cannot be too good: in the best case scenario it is \( O(h^{1-q} \times \mu^r h^r) \) while the principal part of asymptotics is of magnitude \( O(h^{-q} \times \mu^r h^r) \). Multiplying by the number of operators we get remainder estimate \( O(h^{1-d}) \) (in the best case) and the principal part \( \asymp h^{-d} \).

We got another explanation of the difference between full-rank and not-full-rank cases.
3.3 Next terms

Well, only “main part” of $H$ is of the form (25) and skipping $\mu^{-1} \Phi^{1/2} x'$ in $V$ leads to the approximation error in operator of magnitude $\mu^{-1}$, which is normally too large.
3.3 Next terms

Well, only “main part” of $H$ is of the form (25) and skipping $\mu^{-1}\Phi^{\frac{1}{2}}x'$ in $V$ leads to the approximation error in operator of magnitude $\mu^{-1}$, which is normally too large.

The obvious approach would be to decompose $V(\mu^{-1}\Phi^{\frac{1}{2}}x' + \Phi x'', \mu^{-1} h D'', x''')$ into Taylor series with respect to $x'$, with the small error in the smooth case and with an error $O(\mu^{-l} |\log \mu|^{-\sigma})$ in non-smooth case; then $\varepsilon = C \mu^{-1}$ would be the natural choice.
To get rid of linear with respect to $x'$ terms we multiply operator (24) by $e^{-i\mu h^{-1}L}$ and $e^{i\mu h^{-1}L}$ from the left and right respectively with $L = L(x'', x''', \mu^{-1}hD''; x', \mu hD')$. Then operator will be replaced by

$$H + i\mu^{-1}h^{-1}[H^0, L] + \ldots$$

where $H^0 = \sum_{1 \leq j \leq r} f_j (\mu^2 h^2 D_j^2 + x_j^2)$ and we can find $L = \sum_{1 \leq j \leq r} L_j (x'', x''', \mu^{-1}hD'')\mu hD_j$ such that in $H + \mu^{-1}h^{-1}[H^0, L]$ linear terms disappear.
To get rid of linear with respect to $x'$ terms we multiply operator (24) by $e^{-i\mu^{-1}h^{-1}L}$ and $e^{i\mu^{-1}h^{-1}L}$ from the left and right respectively with $L = L(x''', x''', \mu^{-1}hD''; x', \mu hD')$. Then operator will be replaced by

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where $H^0 = \sum_{1 \leq j \leq r} f_j (\mu^2 h^2 D_j^2 + x_j^2)$ and we can find $L = \sum_{1 \leq j \leq r} L_j (x''', x''', \mu^{-1}hD'') \mu h D_j$ such that in $H + \mu^{-1}h^{-1}[H^0, L]$ linear terms disappear.

Now we have an error $O(\mu^{-l} |\log \mu|^{-\sigma} + \mu^{-2})$ which is sufficiently small unless $q = 0$ (because for $q \geq 1$ we consider only inner zone and we are looking for the less strong remainder estimate) and $(l, \sigma) \succ (2, 0)$. 
In the latter case we should take care of terms of magnitude $\mu^{-2}$ which are either linear with respect to $x', \mu^{-1}hD'$ or quadratic. Multiplying by $e^{-i\mu^{-2}h^{-1}L}$ and $e^{i\mu^{-2}h^{-1}L}$ from the left and right respectively we can get rid of all linear and some quadratic terms: we are left with

$$\sum_j b_j(x'', x''', \mu^{-1}hD'', hD''')(\mu^2 h^2 D_j^2 + x_j^2) +$$  \hspace{1cm} (28)

$$\sum_{j \neq k, f_j = f_k} \left( b'_{jk}(x'', x''', \mu^{-1}hD'', hD''')(\mu^2 h^2 D_j D_k + x_j x_k) + b''_{jk}(x'', x''', \mu^{-1}hD'', hD''')\mu h(x_k D_j - x_j D_k) \right)$$  \hspace{1cm} (29)

Terms (28) are always there but terms (29) are due to second-order resonances: $f_j = f_k$ with $j \neq k$. 
If we continue, third-order resonances $f_j = f_k + f_m$ and $f_j = 2f_k$ will generate non-removable terms etc.
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We can live with terms (29) and others non-removable terms. For operator (2) situation is way worse!
If we continue, third-order resonances $f_j = f_k + f_m$ and $f_j = 2f_k$ will generate non-removable terms etc.

We can live with terms (29) and others non-removable terms. For operator (2) situation is way worse!

**Choice of $\varepsilon$.** So we choose $\varepsilon = C\mu^{-1}$ as $\mu \leq h^{-1}|\log \mu|^{-1}$ and $\varepsilon = C(\mu^{-1}h|\log \mu|)^{1/2}$ otherwise.
3.4 Special case: \( d = 2, 3 \)

In this case we can take smaller \( \varepsilon = C(\mu^{-1}h|\log \mu|)^{\frac{1}{2}} \) even if \( \mu \leq h^{-1}|\log \mu|^{-1} \).
3.4 Special case: \( d = 2, 3 \)

In this case we can take smaller \( \varepsilon = C(\mu^{-1}h|\log \mu|)^{\frac{1}{2}} \) even if \( \mu \leq h^{-1}|\log \mu|^{-1} \). We do not decompose into Taylor series, but just are trying to remove as much as possible.
3.4 Special case: $d = 2, 3$

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$$\int_0^{2\pi} K(\rho \cos \phi, \rho \sin \phi, .)d\phi = 0$$

and we are left with terms which do not depend on $\phi$:

$$W(x'', x''', \mu^{-1} hD'', hD'''; (h^2 D_1^2 + \mu^{-2} h_1^2)^{\frac{1}{2}}).$$
3.4 Special case: \( d = 2, 3 \)

In this case we can take smaller \( \varepsilon = C(\mu^{-1} h |\log \mu|)^{\frac{1}{2}} \) even if \( \mu \leq h^{-1} |\log \mu|^{-1} \). We do not decompose into Taylor series, but just are trying to remove as much as possible. Note that for \( r = 1 \) \( i \mu^{-1} h^{-1} [H^0, L] \) is an operator with the symbol \( \partial_\phi L(x, \xi) \) where \((\phi, \rho)\) are polar coordinates in \((x_1, \xi_1)\)-plane. Then we can remove any term with symbol \( K \) such that

\[
\int_0^{2\pi} K(\rho \cos \phi, \rho \sin \phi, .) d\phi = 0 \quad \text{and we are left with terms which do not depend on } \phi:
\]

\[
W \left( x'', x''', \mu^{-1} hD'', hD'''; (h^2 D_1^2 + \mu^{-2} h_1^2)^{\frac{1}{2}} \right).
\]

We can handle them but all construction is really complicated. This not work for \( r \geq 2 \).
3.5 Superstrong Magnetic Field

Everything works nicely as

$$\mu \geq \epsilon h^{-1}.$$  \hfill (30)
3.5 Superstrong Magnetic Field

Everything works nicely as

$$
\mu \geq \epsilon h^{-1}.
$$

(30)

To make the problem reasonable instead of $V \in C^{l,\sigma}$ we need to assume for $\mu \gg h^{-1}$ that

(H1) $\exists \alpha$ such that $V = - \sum_j (2\alpha_j + 1)\mu h f_j + V'$ with $V' \in C^{l,\sigma}$ as $q = 0$.

(H2) $V = - \sum_j \mu h f_j + V'$ with $V' \in C^{l,\sigma}$ as $q \geq 1$. 
Now after we reduced operator to quasicanonical form, we can derive sharp spectral asymptotics. In principal, we have a family \( \{H_\alpha\}_{\alpha \in J} \) of operators, but there could be non-diagonal terms of magnitude \( O(\mu^{-2}) \). Still I was able to tackle them.

Note that \( \#J \asymp (\mu h)^{-r} \) as \( \mu h \leq 1 \) and \( \#J \asymp 1 \) as \( \mu h \geq 1 \) under assumption (H1) or (H2) (case when \( \#J = 0 \) is not interesting).

### 4.1 Full-Rank case

In this case we have a family of \( \mu^{-1} h \)-pdos \( V_\alpha(x'', \mu^{-1} hD'') \). We must assume that they are non-degenerated on level \( \tau \) which
means condition (7) for $\mu h \leq \epsilon_0$ and weaker condition

$$|\nabla V| + \min_{\alpha \in \mathbb{Z}^+_r} |V + \sum_j (2\alpha_j + 1)\mu h f_j - \tau| \geq \epsilon$$

(31)

for $\mu h \geq \epsilon_0$. 
means condition (7) for $\mu h \leq \epsilon_0$ and weaker condition

$$|\nabla V| + \min_{\alpha \in \mathbb{Z}^+} |V + \sum_j (2\alpha_j + 1)\mu h f_j - \tau| \geq \epsilon$$

(31)

for $\mu h \geq \epsilon_0$.

Then the contribution of each of them to remainder estimate is $O(\mu^{r-1} h^{1-r})$ and to the approximation error $O(\epsilon^l |\log \epsilon|^{-\sigma} \mu^{r} h^{-r})$ and there are $(\mu h)^{-r} + 1$ of them. This and the choice of $\epsilon$ results in the following main statements:
Theorem 5  Let $d = 2r$.

(i) Let $\mu h \leq \epsilon_0$ and condition (7) hold. Then for

$$\mu \geq h^{-\frac{1}{2}}|\log h|^{-\frac{1}{2}}$$

(32)

the following estimate holds

$$R = | \int (e(x, x, \tau) - E^\text{MW}(x, \tau))\psi(x) \, dx | \leq$$

$$C \mu^{-1} h^{1-d} + C \mu^{-l} |\log h|^{-\sigma} h^{-d} + C \mu^{-\frac{l}{2}} h^{-d+\frac{l}{2}} |\log h|^{\frac{l}{2}-\sigma}.$$  (33)

(ii) Let $\mu h \geq \epsilon_0$ and conditions (H1), (31) hold. Then

$$R \leq C \mu^{r-1} h^{1-r} + C \mu^{r-\frac{l}{2}} h^{-r+\frac{l}{2}} |\log \mu|^{\frac{l}{2}-\sigma}.$$  (34)
One can improve this theorem as \( d = 2 \) because of the better choice of \( \varepsilon \) for \( \mu \leq h^{-1} |\log h|^{-1} \):

**Theorem 6** Let \( d = 2 \).

(i) Let \( \mu h \leq \epsilon_0 \) and condition (7) hold. Then for

\[
\begin{align*}
 h^{-\frac{1}{3}} |\log h|^{-\frac{1}{3}} & \leq \mu \leq h^{-1} |\log h|^{-1} \\
\end{align*}
\] (35)

the following estimate holds

\[
R' = | \int (e(x, x, \tau) - E^{MW}(x, \tau) - E^{MW}_{corr}(x, \tau)) \psi(x) \, dx | \leq C \mu^{-1} h^{1-d} + C \mu^{-\frac{l}{2}} h^{-d+\frac{l}{2}} |\log h|^{\frac{l}{2}-\sigma}. \] (36)
Here $E_{\text{corr}}^{\text{MW}}(x, \tau)$ is a correction arising from the fact that the difference $\int_C V \, d\phi$ where $C$ is a magnetron orbit cannot be calculated with the good precision by Taylor decomposition of $V$ in its center if $V$ is not smooth. The formula for $E_{\text{corr}}^{\text{MW}}$ is rather complicated.

Combining with Theorem 1 we arrive to

**Corollary 7** (i) For $d = 2r$, $\mu h \leq \epsilon_0$ and condition (7) fulfilled sharp remainder estimate $O(\mu^{-1} h^{1-d})$ is achieved as $V \in C^{3, \frac{3}{2}}$ in the general case and $V \in C^{2, 1}$ as $d = 2$.

(ii) For $d = 2r$, $\mu h \geq \epsilon_0$ and conditions (H1), (31) fulfilled sharp remainder estimate $O(\mu^{r-1} h^{1-r})$ is achieved as $V \in C^{2, 1}$. 
4.2 Not-Full-Rank case

As I mentioned, we have now essentially a family of $q$-dimensional Schrödinger operators

$$\sum_{1 \leq j \leq q} h^2 D_j^2 + V_\alpha(x'', x''', \mu^{-1} h D''')$$

with potential which are $r$-dimensional $\mu^{-1} h$-pdos.
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Assume first that non-degeneracy condition (7) holds as $\mu h \leq \epsilon_0$ and conditions (H2), (31) hold as $\mu h \leq \epsilon_0$. Then each operator gives us remainder estimate $O(\mu^r h^{1-r-q})$. 
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$$
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$$

with potential which are $r$-dimensional $\mu^{-1}h$-pdos. Assume first that non-degeneracy condition (7) holds as $\mu h \leq \epsilon_0$ and conditions (H2), (31) hold as $\mu h \leq \epsilon_0$. Then each operator gives us remainder estimate $O(\mu^r h^{1-r-q})$. Actually, the same would be true as either $q \geq 3$ or $q = 2$, $(l, \sigma) \succeq (2, 0)$ without any non-degeneracy condition. Otherwise estimate would be not as good. While arguments are instructive, we have no time for
them now.
Further, there are $\asymp ((\mu h)^{-r} + 1)$ of these operators; so the total remainder estimate is $O(h^{1-d} + \mu^r h^{1-r})$. 
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Further, there are $\asymp ((\mu h)^{-r} + 1)$ of these operators; so the total remainder estimate is $O(h^{1-d} + \mu^r h^{1-r})$. For $q \geq 2$ contribution of zone $\{ |\xi'''| \leq \bar{\rho} \}$ to the approximation error is $O(\bar{\rho}^{q-2} \varepsilon^{-l} |\log \varepsilon|^{-\sigma})$; for $q = 1$ and respective-non-degeneracy condition fulfilled the answer is $O(\varepsilon^{-l} |\log \varepsilon|^{-\sigma})$. 
Further, there are $\asymp ((\mu h)^{-r} + 1)$ of these operators; so the total remainder estimate is $O(h^{1-d} + \mu^r h^{1-r})$. For $q \geq 2$ contribution of zone $\{|\xi'''| \leq \bar{\varrho}\}$ to the approximation error is $O(\bar{\varrho}^{q-2}\varepsilon^{-l}|\log \varepsilon|^{-\sigma})$; for $q = 1$ and respective-non-degeneracy condition fulfilled the answer is $O(\varepsilon^{-l}|\log \varepsilon|^{-\sigma})$.

Actually, there should be extra mollification to get above-remainder estimate but it does not change the final answer.
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Actually, there should be extra mollification to get above-remainder estimate but it does not change the final answer.

So calculating approximation error (and combining with Theorems 3, 4) we get:
Theorem 8  Let $q \geq 1$. Let either one of the following assumptions be fulfilled:

(a) $q \geq 3$ and $V \in C^{1,1}$,

(b) $q = 2$ and $V \in C^{2,1}$,

or non-degeneracy condition (7) and one of the following assumptions be fulfilled:

(c) $q = 2$, $V \in C^{1,1}$,

(d) $q = 1$, $V \in C^{3/2,1/2}$,

Then for $\mu h \leq \epsilon_0$

$$R = | \int (e(x,x,\tau) - E^{MW}(x,\tau))\psi(x) \, dx | \leq C h^{1-d}. \quad (37)$$

Because $d = 3 \implies r = 1$ and special choice of $\epsilon$ we have
Theorem 9 \( \text{Let } d = 3, \ V \in C^{1,2} \) and non-degeneracy condition (7) be fulfilled. Then for \( \mu h \leq \epsilon_0 \)

\[
R' = \left| \int \left( e(x, x, \tau) - E^\text{MW}(x, \tau) \right) - E^\text{MW}_{\text{corr}} (x, \tau) \right| \psi(x) dx \right| \leq C h^{1-d}. \quad (38)
\]
Theorem 10  Let $q \geq 1$, $\mu h \geq \epsilon_0$ and (H2) hold. Let 

either one of the following assumptions be fulfilled:

(a) $q \geq 3$ and $V \in C^{1,1}$,

(b) $q = 2$ and $V \in C^{2,1}$,

or $q = 1, 2$, $V \in C^{1,1}$ and non-degeneracy condition (31) be fulfilled.

Then

$$R \leq C \mu^r h^{1-d+r}.$$  (39)
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