Complete Spectral Asymptotics for Periodic and Almost Periodic Perturbations of Constant Coefficients Operators and Bethe-Sommerfeld Conjecture in Semiclassical Settings

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Introduction

This talk represents two works. The first of them is inspired by several remarkable papers of L. Parnovski and R. Shterenberg [PSh1, PSh2, PS3], S. Morozov, L. Parnovski and R. Shterenberg [MPSh] and earlier papers by A. Sobolev [So1, So2]. I wanted to understand the approach of the authors and, combining their ideas with my own approach, generalize their results.

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In these papers the complete asymptotic expansion of the integrated density of states $\mathsf{N}(\lambda)$ for operators $\Delta + V$ was derived as $\lambda \to +\infty$; here Δ is a positive Laplacian and V is a periodic or almost periodic potential (satisfying certain conditions). In [MPSh] more general operators were considered.

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The second work is inspired by a paper [PSo] by L. Parnovski and A. Sobolev, in which a classical Bethe-Sommerfeld conjecture was proven for operators $\Delta^m + B(x, D)$ with operator B of order $< 2m$.

In the first work I borrowed from these papers Conditions (A)–(D) and the *special gauge transformation* and added the *non-stationary semiclassical Schrödinger operator method*–[Ivr1] and extremely long propagation of singularities. I believe that this is a simpler and more powerful approach. Also, in contrast to those papers I consider more general semiclassical asymptotics.

In the first work I borrowed from these papers Conditions (A)–(D) and the *special gauge transformation* and added the *non-stationary semiclassical Schrödinger operator method*–[Ivr1] and extremely long propagation of singularities. I believe that this is a simpler and more powerful approach. Also, in contrast to those papers I consider more general semiclassical asymptotics.

Similar ideas were used in the second work with the classification of resonant points borrowed from the same papers.

Consider a scalar self-adjoint h-pseudo-differential operator

$$A_h := A(x, hD) = A^0(hD) + \varepsilon B(x, hD), \tag{1}$$

in $\mathbb{R}^d$ where $A^0(\xi)$ is smooth and elliptic

$$|D_\xi^\beta A(\xi)| \le c_\beta(|\xi| + 1)^m \qquad \forall \xi \in \mathbb{R}^d \ \forall \beta, \tag{2}$$

and

$$A(\xi) \ge c_0|\xi|^m - C_0 \qquad \forall \xi \in \mathbb{R}^d \tag{3}$$

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and on energy level τ satisfies *microhyperbolicity* and *strong convexity* conditions:

$$|A^0(\xi) - \tau| + |\nabla_\xi A^0(\xi)| \ge \epsilon_0 \tag{4}$$

and

$$\pm \sum_{j,k} A^0_{\xi_j\xi_k}(\xi)\eta_j\eta_k \ge \epsilon_0|\eta|^2 \quad \forall \xi\colon A^0(\xi)=\tau \ \ \forall \eta\colon \sum_j A^0_{\xi_j}(\xi)\eta_j = 0. \tag{5}$$

Meanwhile $B(x,\xi)$ is smooth

$$|D_x^\alpha D_\xi^\beta B(x,\xi)| \le c_{\alpha\beta}(|\xi|+1)^m \qquad \forall \alpha,\beta \tag{6}$$

and almost periodic

$$B(x,\xi)=\sum_{\theta\in\Theta} b_\theta(\xi)e^{i\langle\theta,x\rangle} \tag{7}$$

with discrete (i.e. without any accumulation points) *frequency set* Θ, $\varepsilon>0$ is a small parameter: $h\le\varepsilon\le h^{\varkappa}$ with $\varkappa>0$.

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Then it is semibounded from below. Let $e_{h,\varepsilon}(x,y,\lambda)$ be the Schwartz kernel of its spectral projector $E(\lambda) = \uptheta(\lambda - A)$.

First main theorem

Theorem 1.

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Let Conditions (A)–(D) below be fulfilled (they are fulfilled automatically if B is periodic, i.e. Θ is a non-degenerate lattice). Then for $|\tau - \lambda| < \epsilon$, $\varepsilon \le h^{\varkappa}$, $\varkappa > 0$

$$e_{h,\varepsilon}(x,x,\tau) \sim \sum_{n\ge 0} \kappa_n(x,\tau;\varepsilon) h^{-d+n} \qquad \text{as} \quad h \to +0. \tag{8}$$

Corollary 2.

In the framework of Theorem 1

$$\mathsf{N}_{h,\varepsilon}(\tau) \sim \sum_{n\geq 0} \bar{\kappa}_n(\tau;\varepsilon) h^{-d+n} \qquad \textit{as}\ \ h \to +0.. \tag{9}$$

Here

$$\mathsf{N}_h(\lambda) = \mathsf{M}[e(x,x,\lambda)] \ := \lim_{\ell\to\infty} (\mathrm{mes}(\ell X))^{-1} \int_{\ell X} e(x,x,\lambda)\,dx, \tag{10}$$

where $0 \in X$ is an open domain in $\mathbb{R}^d$. The latter expression in the cases we are interested in does not depend on X and is called *Integrated Density of States*.

Conditions (A)-(D)

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Condition (A).

For each $\theta_1, \ldots, \theta_d \in \Theta$ either $\theta_1, \ldots, \theta_d$ are linearly independent over $\mathbb{R}$ or they linearly dependent over $\mathbb{Z}$.

Assume also that

Condition (B).

For any arbitrarily large L and for any sufficiently large real number r there are a finite symmetric about 0 set $\Theta' := \Theta'_{(L,r)} \subset (\Theta \cap \mathrm{B}(0,r))$ (with $\mathrm{B}(\xi, r)$ the ball of the radius r and center ξ) and a "cut-off" coefficients $b'_\theta := b'_{\theta,(L,r)}$, such that

$$B' := B'_{(L,r)}(x,\xi) := \sum_{\theta \in \Theta'} b'_\theta(\xi) e^{i\langle \theta, x\rangle} \tag{11}$$

satisfies

$$\|D_x^\alpha D_\xi^\beta \bigl(B - B'\bigr)\|_{\mathscr{L}^\infty} \le r^{-L}(|\xi|+1)^m \qquad \forall \alpha,\beta\colon |\alpha| \le L, |\beta| \le L. \tag{12}$$

Remark 1.

1. Then

$$|D_\xi^\beta b_\theta| = O(|\theta|^{-\infty}(|\xi|+1)^m) \qquad \text{as} \quad |\theta|\to\infty \tag{13}$$

and

$$|D_\xi^\beta (b_\theta - b'_\theta)| = O(r^{-\infty}(|\xi|+1)^m). \tag{14}$$

Indeed, one suffices to observe that $b_\theta(\xi)=\mathrm{M}(B(x,\xi)e^{-i\langle\theta,x\rangle})$ etc.

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Indeed, one suffices to observe that $b_\theta(\xi) = \mathrm{M}(B(x,\xi)e^{-i\langle\theta,x\rangle})$ etc.

2. On the other hand, under additional assumption

$$\#\{\theta\in\Theta,\, |\theta|\le r\} = O(r^p) \qquad \text{as} \quad r\to\infty \tag{15}$$

for some p, (13) implies Condition (B) with $\Theta'_{(L,r)} := \Theta\cap \mathrm{B}(0,r)$. However we will need $\Theta'_{(L,r)}$ in the next condition.

Remark 1 (Continued).

3. We need only to estimate the operator norm of the difference between $B(x,hD)$ and $B'(x,hD)$ (from $\mathscr{H}^m$ to $\mathscr{L}^2$); therefore for differential operators we can weaken (12).

Remark 1 (Continued).

3. We need only to estimate the operator norm of the difference between $B(x, hD)$ and $B'(x, hD)$ (from $\mathscr{H}^m$ to $\mathscr{L}^2$); therefore for differential operators we can weaken (12).
4. While Condition (B) is Condition B of [PS3], adopted to our case, Condition (A) and Conditions (C), (D) below are borrowed without any modifications (except changing notations).

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$$\Theta'_K := \sum_{1\le i\le K} \Theta. \tag{16}$$

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We say that $\mathfrak{V}$ is a *quasi-lattice subspace* of dimension q, if $\mathfrak{V}$ is a linear span of q linear independent vectors $\theta_1,\dots,\theta_q \in \Theta'_K \setminus 0$. Obviously, the zero space is a quasi-lattice subspace of dimension 0 and $\mathbb{R}^d$ is a quasi-lattice subspace of dimension d.

We denote by $\mathcal{V}_q$ the collection of all quasi-lattice subspaces of dimension q and also $\mathcal{V} := \bigcup_{q\ge 0} \mathcal{V}_q$.

Consider $\mathfrak{V},\mathfrak{U}\in\mathcal{V}$. We say that these subspaces are *strongly distinct*, if neither of them is a subspace of the other one. Next, let $\widehat{(\mathfrak{V},\mathfrak{U})}\in[0,\pi/2]$ be the angle between them, i.e. the angle between $\mathfrak{V}\ominus\mathfrak{W}$ and $\mathfrak{U}\ominus\mathfrak{W}$, $\mathfrak{W}=\mathfrak{U}\cap\mathfrak{V}$. This angle is positive iff $\mathfrak{V}$ and $\mathfrak{U}$ are strongly distinct.

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Condition (C).

For each fixed L and K the sets $\Theta'_{(L,r)}$ satisfying (11) and (12) can be chosen in such a way that for sufficiently large r we have

$$S(r) = S(\Theta'_K) := \inf_{\mathfrak{V}, \mathfrak{U} \in \mathcal{V}} \sin(\widehat{(\mathfrak{V}, \mathfrak{U})}) \geq r^{-1} \tag{17}$$

and

$$R(r) := \inf_{\theta \in \Theta'_K \setminus 0} |\theta| \geq r^{-1}, \tag{18}$$

where the implied constant (how large should r be) depends on L and K.

Let $\mathfrak{V}$ be the span of $\theta_1, \dots, \theta_q \in \Theta'_K \setminus 0$. Then due to Condition (A) each element of the set $\Theta'_K \cap \mathfrak{V}$ is a linear combination of $\theta_1, \dots, \theta_q$ with rational coefficients. Since the set $\Theta'_K \cap \mathfrak{V}$ is finite, this implies that the set $\Theta'_\infty \cap \mathfrak{V}$ is discrete and is, therefore, a lattice in $\mathfrak{V}$. We denote this lattice by $\Gamma(r; \mathfrak{V})$.

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Our final condition states that this lattice cannot be too dense.

Condition (D).

We can choose $\Theta'_{(L;r)}$, satisfying Conditions (B) and (C) in such a way that for sufficiently large r and for each $\mathfrak{V} \in \mathcal{V}$, $\mathfrak{V} \neq \mathbb{R}^d$, we have

$$\operatorname{vol}(\mathfrak{V}/\Gamma(r;\mathfrak{V})) \geq r^{-1}. \tag{19}$$

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2. Further, if Θ is a finite set and Condition (A) is fulfilled, then $\Theta_\infty := \bigcup_{K\geq 1} \Theta_K$ is a lattice and Conditions (B)–(D) are fulfilled.
3. Furthermore, the same is true, if Θ is an arithmetic sum of a finite set and a lattice.

Assume now that B is periodic with respect to non-degenerate lattice Γ:

$$A(x+\mathsf{y},\xi)=A(x,\xi)\qquad \forall x\in\mathbb{R}^n\quad \forall \mathsf{y}\in\Gamma. \tag{20}$$

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Let us denote by Γ^* the *dual lattice*:

$$\gamma\in\Gamma^* \iff \langle\gamma,\mathsf{y}\rangle\in 2\pi\mathbb{Z} \quad \forall \mathsf{y}\in\Gamma; \tag{21}$$

since we use Γ^* and it's elements in the paper much more often, than Γ and it's elements, it is more convenient for us to reserve notation γ for elements of Γ^*.

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Also let $\mathcal{O}=\mathbb{R}^d/\Gamma$ and $\mathcal{O}^*=\mathbb{R}^d/\Gamma^*$ be *fundamental domains*; we identify them with domains in $\mathbb{R}^d$.

It is well-known that $\operatorname{Spec}(A_h)$ has a *band-structure*. Namely, consider in $\mathscr{L}^2(\mathcal{O})$ operator $A_h(\xi) = A(x, hD)$ with the *quasi-periodic boundary condition*:

$$u(x + \mathsf{y}) = e^{i\langle \mathsf{y},\xi\rangle} u(x) \qquad \forall x \in \mathcal{O} \quad \forall \mathsf{y} \in \Gamma \tag{22}$$

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and depends on ξ continuously. Further,

$$\operatorname{Spec}(A_h) = \bigcup_{\xi\in\mathcal{O}^*} \operatorname{Spec}(A_h(\xi)) =: \bigcup_n \Lambda_{n,h}, \tag{24}$$

with the *spectral bands* $\Lambda_{n,h} := \bigcup_{\xi\in\mathcal{O}^*}\{\lambda_{n,h}(\xi)\}$.

One can prove that the with of the spectral band near energy level τ is $O(h)$. Spectral bands could overlap but they also could leave uncovered intervals, called *spectral gaps*.

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It follows from Theorem 1 that in our assumptions the width of the spectral gaps near energy level τ is $O(h^\infty)$. Bethe-Sommerfeld conjecture in the semiclassical settings claims that there are no spectral gaps near energy level τ (in the corresponding assumptions, which include $d \geq 2$).

Second main theorem

Theorem 3.

Let $d \geq 2$ and let operator A_h be given by (1) with $\varepsilon = O(h^{\varkappa})$ with arbitrary $\varkappa > 0$ and with $A^0_h = A^0(hD)$ satisfying (2)–(5) and $B(x,\xi)$ satisfying (7) and (8) with $\Theta = \Gamma^*$ where Γ is a non-degenerate lattice of periods.

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Condition (E).

For every $\eta \in \Sigma_\tau$, $\eta \ne \xi$, such that $\nabla_\eta A^0(\eta)$ is parallel to $\nabla_\xi A^0(\xi)$ (we call η *antipodal point*) Σ_τ, intersected with some vicinity of η and shifted by $(\xi - \eta)$, coincides in the vicinity of ξ with $\{\zeta\colon \zeta_k = g(\zeta_{\hat{k}}\}$ and Σ_τ coincides in the vicinity of ξ with $\{\zeta\colon \zeta_k = f(\zeta_{\hat{k}}\}$ and $\nabla^\alpha (f-g)(0) \ne 0$ for some $\alpha\colon |\alpha| = 2$.

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Then $\mathrm{Spec}(A_h) \supset [\tau - \epsilon, \tau + \epsilon]$ *for sufficiently small* $\epsilon > 0$.

Remark 3.

1. If Σ_τ is strongly convex and connected then for every $\xi \in \Sigma_\tau$ there exists exactly one antipodal point $\eta \in \Sigma_\tau$ and $\nabla_\eta A^0(\eta) \upharpoonleft\downharpoonright \nabla_\xi A^0(\xi)$ and Condition (E) is fulfilled.

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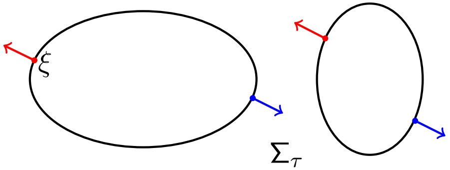
2. If Σ_τ is is strongly convex and consists of p connected components, then the set $\mathfrak{Z}(\xi)=\{\eta\in\Sigma_\tau,\eta\ne\xi\colon \nabla_\eta A^0(\eta)\parallel\nabla_\xi A^0(\xi)\}$ contains exactly $2p-1$ elements, and for p antipodal points $\nabla_\eta A^0(\eta)\upharpoonleft\!\downharpoonright\nabla_\xi A^0(\xi)$ and Condition (E) is fulfilled for sure, while for $(p-1)$ of them $\nabla_\eta A^0(\eta)\upharpoonleft\!\upharpoonright\nabla_\xi A^0(\xi)$ and Condition (E) needs to be satisfied.



One needs to understand, how gaps could appear, why they appear if $d = 1$ and why it is not the case if $d \geq 2$. Note that for A_h^0 one can use instead of $\lambda_{n,h}(\xi)$ functions $\lambda^0_{\gamma,h}(\xi) = A^0((h(\gamma + \xi))$.

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Indeed, in the basis of eigenfunctions of $A^0_\xi(hD)$ (consisting of $\exp(i\langle x, \gamma + \xi\rangle))$ perturbation $\varepsilon B(x, hD)$ can contain out-of-diagonal elements $\varepsilon b_{\gamma-\gamma'}(\xi)$ and such identification is possible only if $|\lambda^0_\gamma(\xi) - \lambda^0_{\gamma'}(\xi)|$ is larger than the size of such element.

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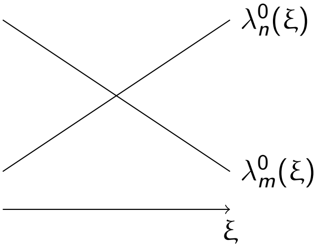
If $d=1$, $A^0(\xi)=\xi^2$ and $\varepsilon\ll h$ and $\tau\asymp 1$, it can happen only if γ' coincides with $-\gamma$ or with one of two adjacent points in Γ^* and $|\xi-\frac{1}{2}(\gamma+\gamma')|=O(\varepsilon h^\infty)$.

One needs to understand, how gaps could appear, why they appear if $d=1$ and why it is not the case if $d\geq 2$. Note that for A^0_h one can use instead of $\lambda_{n,h}(\xi)$ functions $\lambda^0_{\gamma,h}(\xi)=A^0((h(\gamma+\xi))$.

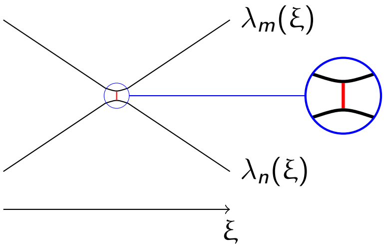
Observe that $\lambda_n(\xi)$ can be identified with some $\lambda^0_\gamma(\xi)$ only locally, if $\lambda^0_\gamma(\xi)$ is sufficiently different from $\lambda^0_{\gamma'}(\xi)$ for any $\gamma'\neq\gamma$.

Indeed, in the basis of eigenfunctions of $A^0_\xi(hD)$ (consisting of $\exp(i\langle x,\gamma+\xi\rangle))$ perturbation $\varepsilon B(x,hD)$ can contain out-of-diagonal elements $\varepsilon b_{\gamma-\gamma'}(\xi)$ and such identification is possible only if $|\lambda^0_\gamma(\xi)-\lambda^0_{\gamma'}(\xi)|$ is larger than the size of such element.

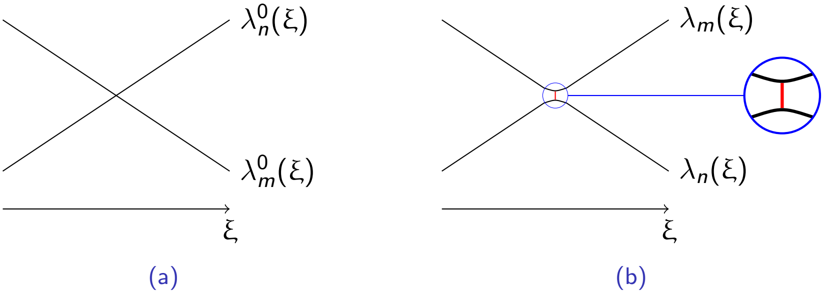
If $d=1$, $A^0(\xi)=\xi^2$ and $\varepsilon\ll h$ and $\tau\asymp 1$, it can happen only if γ' coincides with $-\gamma$ or with one of two adjacent points in Γ^* and $|\xi-\frac{1}{2}(\gamma+\gamma')|=O(\varepsilon h^\infty)$. This exclude from possible values of either $\lambda^0_\gamma(\xi)$ or $\lambda^0_{\gamma'}(\xi)$ the interval of the width $O(h^\infty)$ and on such interval can happen (and really happens for a generic perturbation) the realignment:



(a)



(b)



If $d \ge 2$ the picture becomes more complicated: there are much more opportunities for $\lambda^0_\gamma(\xi)$ and $\lambda^0_{\gamma'}(\xi)$ to become close, even if γ and γ' are not that far away; on the other hand, there is a much more opportunities for us to select $\xi = h(\gamma + \xi) \in \Sigma_\tau$ and then to adjust ξ so that $\xi = h(\gamma + \xi)$ remains on Σ_τ but $\eta = h(\gamma' + \xi)$ moves away from Σ_τ sufficiently far away and then tune-up ξ once again so that $\tau \in \operatorname{Spec}(A_h(\xi))$.

Theorem 3 follows from

Theorem 4.

In the framework of Theorem 3 there exist n and ξ^* *such that* $\lambda_n(\xi^*) = \tau$ *and* $\lambda_n(\xi)$ *covers interval* $[\tau - \upsilon h, \tau + \upsilon h]$ *when* ξ *runs ball* $\mathsf{B}(\xi^*, \upsilon)$ *while* $|\lambda_m(\xi) - \tau| \ge \epsilon \upsilon h$ *for all* $m \ne n$ *and* $\xi \in \mathsf{B}(\xi^*, \upsilon)$.

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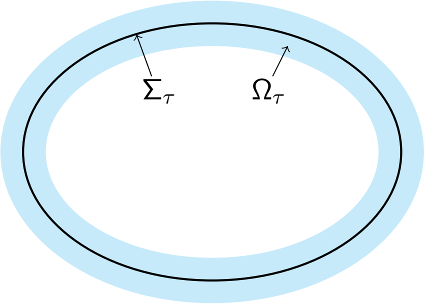
$$\upsilon = \epsilon \begin{cases} h^{(d-1)^2} \min(1,\, \varepsilon^{-3(d-1)/2} h^{(d-1)+\sigma}) & d \geq 3, \\ h \min(|\log h|^{-1},\, \varepsilon^{-3/2} h^{\sigma}) & d = 2 \end{cases} \tag{25}$$

with arbitrarily small exponent $\sigma > 0$*.*

Reduction of operator

Despite Theorems 1 and 4 are of very different nature, their proofs have a common element: reduction of operator to a canonical form in the vicinity of Σ_τ ($0 < \nu$ is very small):

$$\Omega_\tau := \{\xi \colon |A^0(\xi) - \tau| \le C\varepsilon h^{-\nu}\} \tag{26}$$



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$$|\langle \nabla_\xi A^0(\xi), \theta\rangle| \ge \rho \qquad \forall \theta \in \Theta'_K \setminus 0 \tag{27}$$

with $\rho \in [\varepsilon^{1/2} h^{-\delta}, h^{\delta}]$ with arbitrarily small $\delta > 0$.

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Otherwise we call it *resonant*. More precisely

$$\Lambda := \bigcup_{\theta \in \Theta'_K \setminus 0} \Lambda(\theta), \tag{28}$$

where $\Lambda(\theta)$ is the set of ξ, violating (27) for given $\theta \in \Theta'_K \setminus 0$.

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It follows from the microhyperbolicity and strong convexity assumptions (4) and (5) that

Proposition 4.

$\upmu_\tau$-measure ($\upmu_\tau = d\xi : dA^0(\xi)$ is a natural measure on Σ_τ) of $\Lambda \cap \Sigma_\tau$, does not exceed $C_0 r^{d-1}\rho$ and Euclidean measure of $\Lambda \cap \{\xi \colon |A^0(\xi) - \tau| \leq \varsigma\}$ does not exceed $C_0 r^{d-1}\rho\varsigma$.

Classification of resonant points

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Fix $0 < \delta_1 < \ldots < \delta_n$ arbitrarily small and for $\mathfrak{V} \in \mathcal{V}_n$ let us introduce

$$\Lambda(\mathfrak{V}, \rho_n) := \{\xi \in \Omega_\tau \colon |\langle \nabla_\xi A^0(\xi), \theta\rangle| \leq \rho_n |\theta| \quad \forall \theta \in \mathfrak{V}\} \tag{29}$$

with $\rho_n = \varepsilon^{\frac{1}{2}} h^{-\delta_n}$.

We define Ξ_n by induction. First, $\Xi_d = \emptyset$. Assume that we defined $\Xi_d, \ldots, \Xi_{n+1}$. Then we define

$$\Xi_n := \bigcup_{\mathfrak{V}\in\mathcal{V}_n,\ \xi\in\Lambda(\mathfrak{V})\cap\Omega_\tau} (\xi + \mathfrak{V}) \cap \Omega_\tau. \tag{30}$$

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3. *For each $\xi\in\Xi_n\setminus\Xi_{n+1}$ is defined just one $\mathfrak{V}=\mathfrak{V}(\xi)$ such that $\xi\in\xi'+\mathfrak{V}$ for some $\xi'\in\Lambda(\mathfrak{V})$.*

We slightly change definition of Ξ_n: $\xi = h(\gamma + \xi) \in \Xi_{n,\mathrm{new}}$ iff $h\gamma \in \Xi_n$.
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This relation is reflexive, symmetric and transitive.

For $\xi \in \Xi_n$ we define

$$\mathfrak{X}(\xi) = \{\xi' \colon \xi' \cong \xi\}. \tag{31}$$

Then

$$\operatorname{diam}(\mathfrak{X}(\xi)) \le C\rho_{d-1}. \tag{32}$$

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1. *Then there exists a pseudodifferential operator* $P = P(x, hD)$ *such that*

$$\bigl(e^{-i\varepsilon h^{-1}P} A e^{i\varepsilon h^{-1}P} - \mathcal{A}\bigr) Q \equiv 0 \tag{33}$$

with

$$\mathcal{A} = A^0(hD) + \varepsilon B'(hD) + \varepsilon B''(x, hD) \tag{34}$$

modulo operator from $\mathscr{H}^m$ *to* $\mathscr{L}^2$ *with the operator norm* $O(h^M)$ *with* M *arbitrarily large and* $K = K(M, d, \delta)$ *in the definition of non-resonant point provided* $Q = Q(hD)$ *has a symbol, supported in* $\{\xi \colon |A^0(\xi) - \tau| \le 2C\varepsilon h^{-\nu}\}$.

Theorem 5 (continuation).

Here $P(x,hD)$, $B'(hD)$ and $B''(x,hD)$ are operator with Weyl symbols of the same form (7) *albeit such that*

$$|D_\xi^\alpha D_x^\beta P| \le c_{\alpha\beta}\rho^{-1-|\alpha|} \qquad \forall\alpha,\beta, \tag{35}$$

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$$\xi \notin \Lambda(\theta) \implies b''_\theta(\xi) = 0. \tag{37}$$

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In what follows

$$\mathcal{A}^0(hD):=A^0(hD)+\varepsilon B'(hD) \qquad \text{and} \qquad \mathcal{B}:=B''(x,hD). \tag{38}$$

Gauge transformation and proof of Theorem 5

First of all, replace B by $B' := B'_{(L,r)}$ from Condition (B) with $r = h^{-\nu}$, arbitrarily small $\nu > 0$ and $L = 3M/\nu$.

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Consider now the "gauge" transformation $A \mapsto e^{-i\varepsilon h^{-1}P} A e^{i\varepsilon h^{-1}P}$ with h-pseudodifferential operator P. Observe that

$$e^{-i\varepsilon h^{-1}P} A e^{i\varepsilon h^{-1}P} = A - i\varepsilon h^{-1}[P, A] + \sum_{2\le n\le K-1} \frac{1}{n!}(-i\varepsilon h^{-1})^n \operatorname{Ad}_P^n(A)$$
$$+ \int_0^1 \frac{1}{(K-1)!}(1-s)^{K-1}(-i\varepsilon h^{-1})^K e^{-i\varepsilon h^{-1}sP} \operatorname{Ad}_P^K(A) e^{i\varepsilon h^{-1}sP}\, ds, \quad (39)$$

where $\operatorname{Ad}_P^0(A) = A$ and $\operatorname{Ad}_P^{n+1}(A) = [P, \operatorname{Ad}_P^n(A)]$ for $n = 0, 1, \dots$.

Then *formally* we can compensate εB, taking

$$P = \sum_{\theta} ih\bigl(A^0(\xi+\theta h/2) - A^0(\xi-\theta h/2)\bigr)^{-1} b_\theta(\xi) e^{i\langle\theta,x\rangle}, \tag{40}$$

so that

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$$B' = -ih^{-1}[P, B] + \sum_{2\le n\le K-1} \frac{1}{n!}\varepsilon^{n-2}(-ih^{-1})^n \operatorname{Ad}_P^n(A), \tag{42}$$

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New perturbation, again formally, has a magnitude of ε^2. Repeating this process we will make a perturbation negligible.

Remark 6.

However, we need to address the following issues issues:

1. Denominator $h^{-1}\bigl(A^0(\xi+\theta h/2)-A^0(\xi-\theta h/2)\bigr)=\langle \nabla_\xi A^0,\theta\rangle + O(h^{1-\sigma})$ could be small.

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3. We need to prove that the remainder is negligible.
4. This transformation was used in Section 9 of [PS3] (etc); in contrast to these papers we use Weyl quantization instead of pq-quantization, and have therefore $\bigl(A^0(\xi+\theta h/2)-A^0(\xi-\theta h/2)\bigr)$ instead of $\bigl(A^0(\xi+\theta h)-A^0(\xi)\bigr)$.

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then $B' = \varepsilon h^{-1}[P, B]$ has a symbol, satisfying

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Then we can eliminate a perturbation completely, save terms with the frequency 0, both old and new.
On the other hand, if ξ is resonant we can eliminate only frequencies θ which satisfy (27). This concludes the proof of Theorem 5. □

Complete spectral asymptotics

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The main idea beehind the proof of Theorem 1 is a very long range (up to time $T \asymp h^{-M}$) propagation of singularities.

It is known (see Chapter 4 of [Ivr1]) that under microhyperbolicity condition (4) for $|\tau-\lambda|<\epsilon$ the following complete asymptotics holds:

$$F_{t\to h^{-1}\tau}\bigl[\bar{\chi}_T(t)\bigl(Q_{2x}u_h(x,y,t)\,{}^tQ_{1y}\bigr)|_{y=x}\bigr] \sim \sum_{n\ge 0}\kappa'_{n,Q_1,Q_2}(x,\tau)h^{1-d+n}, \qquad (45)$$

where $u_h(x,y,t)$ is the Schwartz kernel of of the *propagator* $e^{ih^{-1}tA}$, $\bar{\chi}\in\mathscr{C}_0^\infty([-1,1])$, $\bar{\chi}(t)=1$ at $[-\frac{1}{2},\frac{1}{2}]$, $\bar{\chi}_T(t)=\bar{\chi}(t/T)$, $T\in[h^{1-\delta},T^*]$, T^* is a small constant here and $Q_j=Q_j(x,hD)$ are h-pseudo-differential operators; we write operators, acting with respect to y on Schwartz kernels to the right of it.

This equality (45) plus Hörmander's Tauberian theorem imply the remainder estimates $O(h^{1-d}T^{*\,1})$ for $\bigl(Q_{2x}e_h(x,y,\tau)\,{}^tQ_{1y}\bigr)|_{x=y}$.

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Indeed, one can prove, that for such time $\xi(t)$ is confined to $C\epsilon_1$-vicinity of $\xi(0)$. One needs to understand this claim *in the sense of quantum mechanics* (or microlocal analysis):

$$F_{t\to h^{-1}\tau}\bigl[\bar\chi_T(t)Q_1(hD)e^{ih^{-1}tA}Q_2(hD)\bigr]=O(h^M) \tag{46}$$

provided $\mathrm{dist}(\mathrm{supp}(Q_1),\ \mathrm{supp}(Q_2))\ge C\varepsilon T$.

Then x propagates for this time approximately in the direction of $\nabla_\xi A^0(\xi)|_{\xi=\xi(0)}$, again in the sense of quantum mechanics:

$$F_{t\to h^{-1}\tau}\bigl[\chi_T(\pm t)\psi_1(x)e^{ih^{-1}tA}\psi_2(x)Q(hD)\bigr]\equiv 0 \mod O(h^M), \tag{47}$$

provided Q is supported in ϵ-vicinity of ξ^0, and

$$\mathrm{dist}(\mathrm{supp}(\psi_1),\mathrm{supp}(\psi_2)\pm T\nabla_\xi A^0(\xi^0))\ge C_0\epsilon T,$$

where $\chi\in\mathscr{C}_0^\infty([1-\epsilon,1+\epsilon])$.

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Therefore considering its kernel and making some partitions with respect to x, y, hD_x and hD_y we arrive to

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Then the remainder estimate is $O(h^{1-d}T^{*\,-1})=O(\varepsilon h^{1-d})$.

Long-term propagation

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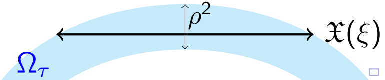

Bethe-Sommerfeld conjecture (sketch of proof)

First of all, let us replace operator $\mathcal{A}$ defined by (34) by operator

$$\mathcal{A}' = \mathcal{A}^0(hD) + \varepsilon\mathcal{B}'(x, hD), \qquad \mathcal{B}'(x, hD) = S(hD)\mathcal{B}S(hD) \tag{49}$$

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From now on $\mathcal{A} := \mathcal{A}'$ and $\mathcal{B} := \mathcal{B}'$.

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Structure of operator $\mathcal{A}$

For $\xi \in \Xi_n \setminus \Xi_{n+1}$ denote by $\mathfrak{H}(\xi)$ the subspace $\mathscr{L}^2(\mathcal{O})$ consisting of functions of the form

$$\sum_{\xi' \in \mathfrak{X}(\xi)} c_{\xi'} e^{i\langle x, \xi' \rangle}. \tag{50}$$

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Let us denote by $\mathcal{A}_\gamma(\xi)$ and $\mathcal{B}_\gamma(\xi)$ restrictions of $\mathcal{A}$ and $\mathcal{B}$ to $\mathfrak{H}(h(\gamma+\xi))$. Here for $n = 0$ we consider Ξ_0 to be the set of all non-resonant points and $\mathfrak{X}(\xi) = \{\xi\}$ for $\xi \in \Xi_0$.

Then due to Propositions 8 and 9 we arrive to

Proposition 10.

1. *For each point* $\lambda \in \operatorname{Spec}(A(\xi)) \cap \{|\lambda - \tau| \le \varepsilon h^{-\delta}\}$ *exists* $\gamma \in \Gamma^*$ *such that* $\xi = h(\gamma + \xi) \in \Omega_\tau$ *and* $\operatorname{dist}(\lambda, \operatorname{Spec}(\mathcal{A}_\gamma(\xi)) \le Ch^M$.

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3. *Further, if* $\lambda\in\operatorname{Spec}(A_\gamma(\xi))\cap\{|\lambda-\tau|\le\varepsilon h^{-\delta}\}$ *is a simple eigenvalue separated from the rest of* $\operatorname{Spec}(A(\xi))$ *by a distance at least* $2h^{M-1}$*, then there exist* γ *and* λ'*, such that for* $\xi=h(\gamma+\xi)$*,* $\lambda'\in\operatorname{Spec}(\mathcal{A}(\xi))\cap\{|\lambda'-\lambda|\le Ch^M\}$*, separated from the rest of* $\operatorname{Spec}(\mathcal{A}_\gamma(\xi))$ *by a distance at least* h^{M-1} *and from* $\bigcup_{\gamma'\in\Gamma^*,\ \gamma'\ne\gamma}\operatorname{Spec}(\mathcal{A}_{\gamma'}(\xi)$ *by a distance at least* h^{M-1} *as well.*

Proposition 10 (continued).

4. *Conversely, if* $\lambda' \in \operatorname{Spec}(\mathcal{A}_\gamma(\xi)) \cap \{|\lambda - \tau| \le \varepsilon h^{-\delta}\}$ *is a simple eigenvalue separated from the rest of* $\operatorname{Spec}(\mathcal{A}_\gamma(\xi))$ *by a distance at least* $2h^{M-1}$*, and also separated from* $\bigcup_{\gamma'\in\Gamma^*,\ \gamma'\ne\gamma} \operatorname{Spec}(\mathcal{A}_{\gamma'}(\xi))$ *by a distance at least* $2h^{M-1}$*, then there exists* $\lambda \in \operatorname{Spec}(A(\xi)) \cap \{|\lambda' - \lambda| \le Ch^M\}$ *separated from the rest of* $\operatorname{Spec}(A(\xi))$ *by a distance at least* h^{M-1}*.*

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In virtue of Proposition 4 we can select $\xi^*_{\mathsf{new}} \in \Sigma_\tau$ such that $|\xi^*_{\mathsf{new}} - \xi^*| \le h^\delta$ and ξ^*_{new} satisfies (27) with $\rho = \gamma := h^\delta$. Here $\delta > 0$ is arbitrarily small and $\nu = \nu(\delta)$.

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Then, according to Theorem 5 we can diagonalize operator in γ-vicinity of ξ^* and there $\rho = \gamma$. Then there

$$|\nabla^\alpha(\mathcal{A}^0 - A^0)| \leq C_\alpha(\varepsilon + \varepsilon^2\rho^{-2-|\alpha|}) \tag{51}$$

and in particular

$$|\nabla^\alpha(\mathcal{A}^0 - A^0)| \leq Ch^\delta \qquad \text{for} \quad |\alpha| \leq 2. \tag{52}$$

Let

$$\Sigma'_\tau = \{\xi\colon \mathcal{A}^0(\xi) = \tau\}. \tag{53}$$

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$$|\lambda_\gamma(\xi) - \lambda_{\gamma^*}(\xi)| \ge \epsilon h^{1+\delta} \qquad \forall \gamma\colon |\gamma - \gamma^*| \le K h^{-\nu} \quad \forall \xi \in \mathcal{O}^*. \tag{54}$$

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$$|\lambda_\gamma(\xi) - \lambda_{\gamma^*}(\xi)| \ge \epsilon h^{1+\delta} \qquad \forall \gamma\colon |\gamma-\gamma^*| \le Kh^{-\nu} \quad \forall \xi \in \mathcal{O}^*. \tag{54}$$

Indeed, $|\gamma-\gamma^*| \le Kh^{-\nu}$ implies that $(\gamma-\gamma^*) \in \Theta'_K$ and then

$$|\langle \nabla \mathcal{A}^0(\xi^*), \gamma-\gamma^*\rangle| \ge \gamma$$

while

$$|\lambda_\gamma(\xi) - \lambda_{\gamma^*}(\xi) - h\langle \nabla \mathcal{A}^0(\xi^*), \gamma-\gamma^*\rangle| \le Ch^{3-3\nu}.$$

Consider other non-resonant points (with $\rho = \varepsilon^{1/2} h^{-\delta}$). Let us determine how $\lambda_\gamma(\xi)$ changes when we change ξ. Due to (52)

$$\delta\lambda_\gamma := \lambda_\gamma(\xi + \delta\xi) - \lambda_\gamma(\xi) = h\langle \nabla A^0(\xi), \delta\xi\rangle + O(h^2|\delta\xi|^2). \tag{55}$$

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Let us take $\delta\xi = t\eta$

$$\ell\colon |\eta| = 1, \qquad \langle \nabla \mathcal{A}^0(\xi^*), \eta\rangle = 0. \tag{56}$$

Then in all non-resonant points ξ the shift will be $\langle \nabla_\xi \mathcal{A}(\xi), \delta\xi\rangle$ with an absolute value $|\langle \nabla_\xi \mathcal{A}(\xi), \eta\rangle| \cdot |t|$.

Case $d = 2$.

Let us start from the easiest case $d = 2$. Without any loss of the generality we assume that ξ^* is strictly inside $\mathcal{O}^*$ (at the distance at least $C\epsilon^*$ from the border). Then there is just one tangent direction η and

$$|\langle \nabla_\xi \mathcal{A}^0(\xi)|_{\xi=h\gamma}, \eta\rangle| \asymp |\sin\varphi(\gamma^*, \gamma)| \asymp h \min_{1\le k\le 2p} |\gamma - \gamma^*_k| \qquad (57)$$

where $\varphi(\gamma^*, \gamma)$ is an angle between $\nabla_\xi \mathcal{A}^0(\xi)|_{\xi=h\gamma^*}$ and $\nabla_\xi \mathcal{A}^0(\xi)|_{\xi=h\gamma}$, and $\xi^*_1, \ldots, \xi^*_{2p-1}$ are antipodal points, and $\xi^*_{2p} = \xi^*$.

Case $d=2$.

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As long as $\min_{1\le k\le 2p}|\gamma-\gamma_k^*|\gtrsim h^{1-\nu}$ we may replace here $\xi=h(\gamma+\xi)$ by $\xi=h\gamma$ and $\mathcal{A}^0$ by $\mathcal{A}$.

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As long as $\min_{1\le k\le 2p} |\gamma - \gamma_k^*| \gtrsim h^{1-\nu}$ we may replace here $\xi = h(\gamma + \xi)$ by $\xi = h\gamma$ and $\mathcal{A}^0$ by $\mathcal{A}$.

In the nonlinear settings to ensure that

$$\lambda_{\gamma^*}(\xi^* + \delta\xi(t)) = \tau \qquad (58)$$

we need to include in $\delta\xi(t)$ a correction: $\delta\xi(t) = t\eta + O(t^2)$ but still

$$\frac{d}{dt}\lambda_\gamma(\xi^* + \delta\xi(t)) \asymp h\langle \nabla_\xi \mathcal{A}(\xi)|_{\xi=h\gamma}, \eta\rangle^{-1}. \qquad (59)$$

Then the set $\mathcal{T}(\xi) := \{t\colon |t| \le \epsilon_0,\ |\mathcal{A}^0(\xi(t)) - \tau| \le \upsilon h\}$ is an interval of the length $\asymp \upsilon$ and then the union of such sets over $\xi = h\gamma + \xi$ with indicated γ does not exceed $R\upsilon$ with

$$R := \sum_{\gamma} |\langle \nabla_\xi \mathcal{A}(\xi)|_{\xi = h\gamma}, \eta\rangle|^{-1}, \qquad (60)$$

where we sum over set

$$\{\gamma |\colon \gamma - \gamma^*| \gtrsim h^{-\nu}\ \&\ |\lambda_\gamma(h\gamma) - \tau| \le 2Ch\}.$$

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One can see easily that $R \asymp h^{-1}|\log h|$. Then, as $R\upsilon \le \epsilon'$ the set $[-\epsilon_0, \epsilon_0] \setminus \bigcup_\gamma \mathcal{T}(h(\gamma + \xi))$ contains an interval of the length $\ell = \upsilon$ and for all t, belonging to this interval,

$$|\lambda_\gamma(h(\gamma + \xi + \delta\xi(t))) - \tau| \ge \epsilon \upsilon h. \tag{61}$$

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Then we need to take $\upsilon = \epsilon R^{-1} = \epsilon h|\log h|^{-1}$ and for $d = 2$ as far as non-resonant are concerned, Theorem 4 is almost proven.

Case $d \geq 3$

In this case we need to be more subtle and to make $(d-1)$ steps. We start from the point $\xi^* = h(\gamma^* + \xi^*)$; again without any loss of the generality we assume that ξ^* is strictly inside $\mathcal{O}^*$ (at the distance at least $C\epsilon^*$ from the border). Then after each step below it still will be the case (with decreasing constant).

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1. On *Step 1* we select $\eta = \eta_1$ and consider only γ such that (57) holds; more precisely, the left-hand expression needs to be greater than the right-hand expression, multiplied by ϵ (one can see easily, that the opposite inequality holds). Then $R \asymp h^{1-d}$ and therefore exists ξ^* such that $\lambda_{\gamma^*}(\xi^*) = \tau$ and $|\lambda_\gamma(\xi^*) - \tau| \geq \epsilon \upsilon_1 h$ with $\upsilon_1 = \epsilon h^{d-1}$ for all γ indicated above.

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2. On *Step 2* we select $\eta = \eta_2$ perpendicular to η_1. To preserve inequality (61) (with smaller constant ϵ) for γ, already covered by Step 1, we need to take $|\delta\xi| \leq \epsilon' \upsilon_1$ and consider $\delta\xi = t\eta_2 + O(t^2)$.

Then the same arguments as before results in inequality (61) with $\upsilon := \upsilon_2 = \epsilon R^{-1}\upsilon_1$ for a new bunch of points. Then for $d = 3$ as far as non-resonant are concerned, Theorem 4 is almost proven.

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3. *Next steps.* Continuing this process, on k-th step we select η_k orthogonal to $\eta_1, \dots, \eta_{k-1}$. Then we get $\upsilon_k = \epsilon R^{-1}\upsilon_{k-1}$ and on the last $(d-1)$-th step we achieve a separation at least $\upsilon_{d-1} = \epsilon R^{1-d}$.

Almost antipodal points

We need to cover points with $|\xi - \xi_k^*| \le h^{1-\kappa}$ for $k = 1, \ldots, 2p-1$ and as we already know for each k (and fixed $\upxi$) there exists no more than one such point $\xi = h(\gamma + \upxi)$ with $|\lambda_\gamma(\upxi) - \tau| \lesssim h^{1+\delta}$.

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We take care of such points during Step 1. Observe that during this step we automatically take care of any point with

$$|\nabla_\xi \mathcal{A}^0(\xi), \eta_1\rangle| \ge \epsilon h, \qquad (62)$$

assuming that $|t| \le \epsilon_0$ with sufficiently small $\epsilon_0 = \epsilon_0(\epsilon)$.

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Let us select η_1 so that on η_1 quadratic forms at points $\xi^*_1, \ldots, \xi^*_{2p-1}$ in Condition (E) are different from one at point ξ^* by at least ϵ_0. Then for each $j = 1, \ldots, 2p-1$ the the measure of the set

$$\{t\colon |t| \le \epsilon_0,\ |\lambda_{\gamma_j}(\xi + \delta\xi(t))| \le \upsilon h\}$$

does not exceed $Ch^{-1}(\upsilon h)^{\frac{1}{2}}$, and then the measure of the union of such sets (by j) also does not exceed it.

Therefore for $\upsilon_1 = \epsilon_1 h^{d-1}$ (for $d \geq 3$) and $\upsilon_1 = \epsilon_1 h|\log h|^{-1}$ (for $d = 2$) with sufficiently small ϵ_1 we can find t: $|t| \leq \epsilon_0$ so that condition (57) is fulfilled for all non-resonant points.

Resonant points

Next on this step we need to separate $\lambda_{\gamma^*}(\xi)$ from all $\lambda_n(\xi)$ (save one, coinciding with it) by the distance at least υh by choosing ξ. We can during the same steps as described in the previous section: let $\lambda_{\gamma,j}(\xi)$ denote eigenvalues of $\mathcal{A}_\gamma(\xi)$ with $j = \#\mathfrak{X}(\gamma h)$.

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Observe that both $\mathcal{A}_\gamma(\xi)$ and $\#\mathfrak{X}(\gamma h)$ depend on the equivalency class $[\gamma]$ of γ rather than on γ itself and that

$$\sum_{[\gamma]} \#\mathfrak{X}(\gamma h) = \sum_{1\le n\le d-1} \#(\Xi_n) = O(h^{1-d+\sigma'} + \varepsilon^{3/2} h^{-d-\sigma}), \tag{63}$$

where on the left $[\gamma]$ runs over all equivalency classes with $\gamma \in \bigcup_{1\le n\le d-1} \Xi_n$.

We also observe that for resonant points

$$|\sin\varphi(\xi,\xi^*)| \ge \epsilon h^\delta \tag{64}$$

and therefore for λ'_γ, which are eigenvalues of $\mathcal{A}^0(h(\gamma+\xi))$ (59) holds and signs are the same for γ in the same block.

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On the other hand,

$$|\frac{d}{dt}\mathcal{B}(h(\gamma+\xi^*+\delta\xi(t))|\leq C\varepsilon h\ll h^{1+\delta'} \tag{65}$$

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Therefore the arguments of each Steps 1, 2 etc extends to resonant points as well. However the number of *new points* to be taken into account on each step is given by the right-hand expression of (63) and therefore R needs to be redefined

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This leads to the final expression (25) for υ. Theorem 4 is proven.

Discussion: differentiability

Remark 11.

1. It also follows from Corollary 2 that

$$\frac{1}{\varsigma}\Bigl[\mathsf{N}_{h,\varepsilon}(\tau+\varsigma)-\mathsf{N}_{h,\varepsilon}(\tau)\Bigr]=\frac{1}{\varsigma}\Bigl[\mathcal{N}_{h,\varepsilon}(\tau+\varsigma)-\mathcal{N}_{h,\varepsilon}(\tau)\Bigr]+O(h^\infty) \quad (67)$$

provided $\varsigma \ge h^M$, where $\mathcal{N}_{h,\varepsilon}(\tau)$ is the right-hand expression of (9).

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   provided $\varsigma\ge h^M$, where $\mathcal{N}_{h,\varepsilon}(\tau)$ is the right-hand expression of (9).

2. The question remains, if (67) holds for smaller ς, in particular, if it holds in $\varsigma\to 0$ limit? If the latter holds, then

$$\frac{\partial}{\partial\tau}\mathsf{N}_{h,\varepsilon}(\tau)=\frac{\partial}{\partial\tau}\mathcal{N}_{h,\varepsilon}(\tau)+O(h^\infty) \quad (68)$$

   and we call the left-hand expression the *density of states*.

Remark 11 (Continued).

3. It definitely is not necessarily true, at least in dimension 1. From now on we consider only asymptotics with respect to $\tau \to +\infty$. Let $A = \Delta + V(x)$ with periodic V. It is well-known that for $d = 1$ and generic periodic V all spectral gaps are open which contradicts to

$$\frac{\partial}{\partial \tau} \mathsf{N}(\tau) = \frac{\partial}{\partial \tau} \mathcal{N}(\tau) + O(\tau^{-\infty}). \tag{69}$$

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4. On the other hand, this objection does not work in case $d \geq 2$ since only several the lowest spectral gaps are open (Bethe-Sommerfeld conjecture, proven in [PSo]).

Remark 11 (End).

5. Further, one can differentiate $e(x,x,\tau^2)$ if $d \geq 2$ and V is compactly supported. Some generalizations are considered in [Ivr4].

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6. Moreover, we can differentiate complete asymptotics of the *Birman-Krein spectral shift function*

$$\zeta(\tau) := \int \bigl(e(x,x,\tau^2) - e^0(x,x,\tau^2)\bigr)\,dx \sim \sum_{n\geq 0} \bar{\kappa}_n \tau^{d-n}, \tag{70}$$

with

$$\bar{\kappa}_n := \int \bigl(\kappa_n(x) - \kappa_n^0\bigr)\,dx, \tag{71}$$

where $e^0(x,y,\tau)$ and κ_n^0 correspond to $A^0 = \Delta$.

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where $e^0(x,y,\tau)$ and κ_n^0 correspond to $A^0=\Delta$. In the case of $A=\Delta$ in the exterior of smooth, compact and non-trapping obstacle and $A^0=\Delta$ in $\mathbb{R}^d$ such asymptotics was derived in [PP].

Discussion: Bethe-Sommerfeld conjecture for almost periodic perturbations

Remark 12.

While both the proof of Bethe-Sommerfeld conjecture and the statement of Theorem 4 rely upon periodicity, the conjecture itself (as stated in Theorem 3) does not.

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While both the proof of Bethe-Sommerfeld conjecture and the statement of Theorem 4 rely upon periodicity, the conjecture itself (as stated in Theorem 3) does not.

It is only natural to try to prove it for almost periodic perturbations.

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