Complete Semiclassical Spectral Asymptotics for Periodic and Almost Periodic Perturbations of Constant Operators

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Introduction

This work is inspired by several remarkable papers of L. Parnovski and R. Shterenberg [PS1, PS2, PS3], S. Morozov, L. Parnovski and R. Shterenberg [MPS] and earlier papers by A. Sobolev [So1, So2]. I wanted to understand the approach of the authors and, combining their ideas with my own approach, generalize their results.

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In these papers the complete asymptotic expansion of the integrated density of states $\mathsf{N}(\lambda)$ for operators $\Delta + V$ was derived as $\lambda \to +\infty$; here Δ is a positive Laplacian and V is a periodic or almost periodic potential (satisfying certain conditions). In [MPS] more general operators were considered.

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Further, in [PS3] the complete asymptotic expansion of $e(x,x,\lambda)$ was derived, where $e(x,y,\lambda)$ is the Schwartz kernel of the spectral projector.

I borrowed from these papers Conditions (A)–(D) and the *special gauge transformation* and added the *non-stationary semiclassical Schrödinger operator method* [Ivr1] and extremely long propagation of singularities. I believe that this is a simpler and more powerful approach. Also, in contrast to those papers I consider more general semiclassical asymptotics.

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Consider a scalar self-adjoint h-pseudo-differential operator $A(x, hD)$ in $\mathbb{R}^d$ with the Weyl symbol $A(x,\xi)$, such that

$$|D_x^\alpha D_\xi^\beta A(x,\xi)| \le c_{\alpha\beta}(|\xi|+1)^m \qquad \forall \alpha,\beta,\ \forall x,\xi \tag{1}$$

and

$$A(x,\xi) \ge c_0|\xi|^m - C_0 \qquad \forall x,\xi. \tag{2}$$

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Then it is semibounded from below. Let $e_h(x,y,\lambda)$ be the Schwartz kernel of its spectral projector $E(\lambda)=\theta(\lambda-A)$.

We are interested in the semiclassical asymptotics of $e_h(x,x,\lambda)$ and

$$\mathsf{N}_h(\lambda) = \mathsf{M}[e(x,x,\lambda)] \;:= \lim_{\ell\to\infty} (\operatorname{mes}(\ell X))^{-1} \int_{\ell X} e(x,x,\lambda)\,dx, \tag{3}$$

where $0 \in X$ is an open domain in $\mathbb{R}^d$. The latter expression in the cases we are interested in does not depend on X and is called *Integrated Density of States*.

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It is well-known that under *ξ-microhyperbolicity condition on the energy level λ*

$$|A(x,\xi,h)-\lambda| + |\nabla_\xi A(x,\xi,h)| \ge \epsilon_0 \tag{4}$$

the following asymptotics holds

$$e_h(x,x,\lambda) = \kappa_0(x,\lambda)h^{-d} + O(h^{1-d}) \qquad \text{as} \quad h\to +0, \tag{5}$$

and therefore

$$\mathsf{N}_h(\lambda) = \bar\kappa_0(\lambda)h^{-d} + O(h^{1-d}), \tag{6}$$

where here and below

$$\bar\kappa_n(\lambda) = \mathsf{M}[\kappa_n(x,\lambda)]. \tag{7}$$

Also it is known (see Chapter 4 of [Ivr1]) that under microhyperbolicity condition (4) for $|\tau-\lambda|<\epsilon$ the following complete asymptotics holds:

$$F_{t\to h^{-1}\tau}\bigl(\bar\chi_T(t)\bigl(Q_{2x}u_h(x,y,t)\,{}^tQ_{1y}\bigr)|_{y=x}\bigr)\sim\sum_{n\ge 0}\kappa'_{n,Q_1,Q_2}(x,\tau)h^{1-d+n},\tag{8}$$

where $u_h(x,y,t)$ is the Schwartz kernel of of the *propagator* $e^{ih^{-1}tA}$, $\bar\chi\in\mathscr{C}_0^\infty([-1,1])$, $\bar\chi(t)=1$ at $[-\frac{1}{2},\frac{1}{2}]$, $T\in[h^{1-\delta},T^*]$, T^* is a small constant here and $Q_j=Q_j(x,hD)$ are h-pseudo-differential operator; we write operators, acting with respect to y on Schwartz kernels to the right of it.

Further,

$$\operatorname{supp}(Q_1)\cap\operatorname{supp}(Q_2)=\emptyset \implies \kappa'_{n,Q_1,Q_2}(x,\tau)=0, \tag{9}$$

where $\operatorname{supp}(Q_j)$ is a support of its symbol $Q_j(x,\xi)$ and

$$\tau\le\tau^*=\inf_{x,\xi} A(x,\xi) \implies \kappa'_{n,Q_1,Q_2}(x,\tau)=0. \tag{10}$$

Let

$$\kappa_{n,Q_1,Q_2}(x,\tau)=\int_{-\infty}^{\tau}\kappa'_{n,Q_1,Q_2}(x,\tau')\,d\tau. \tag{11}$$

In what follows we skip subscripts $Q_j=I$.

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This equality (8) plus Hörmander's Tauberian theorem imply the remainder estimates $O(h^{1-d})$ for $Q_{2x}e_h(x,y,\tau)\,{}^tQ_{1y}|_{x=y}$. On the other hand, if we can improve (8) by increasing T^*, we can improve the remainder estimate to $O(T^{*\,-1}h^{1-d})$ [1].

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Observe that for $A = A(hD)$

$$e_h(x, x, \lambda) = \mathsf{N}_h(\lambda) = \kappa_0(\lambda) h^{-d}. \tag{12}$$

In this paper we consider

$$A(x, hD) = A^0(hD) + \varepsilon B(x, hD), \tag{13}$$

where $A^0(\xi)$ satisfies (1), (2) and (4) and $B(x, \xi)$ satisfies (1) and $\varepsilon > 0$ is a small parameter. Later we assume that $B(x, hD)$ is almost periodic and impose other conditions.

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For operator (13) with $\varepsilon \le \epsilon_0$ the equality (8) holds with $T^* = \epsilon_1 \varepsilon^{-1}$ where ϵ_j are small constants and we assume that $\varepsilon \ge h^M$ for some M. Then the remainder estimate is $O(\varepsilon h^{1-d})$ (Theorem 2.4 of [Ivr3]).

Main Theorem

Consider the main topic of this work where we will use ideas from [PS1, PS2, PS3, MPS]: the case of an almost periodic operator $B(x, hD)$,

$$B(x,\xi) = \sum_{\theta\in\Theta} b_\theta(\xi)e^{i\langle\theta,x\rangle} \tag{14}$$

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Operator B is *quasiperiodic* if Θ is a finite set, *periodic* if Θ is a lattice and *almost periodic* in the general case.

Our goal is to derive (under certain assumptions) complete semiclassical asymptotics:

$$e_{h,\varepsilon}(x,x,\tau) \sim \sum_{n\geq 0} \kappa_{n,\varepsilon} x(x,\tau) h^{-d+n}. \tag{15}$$

In addition to microhyperbolicity condition (4) we assume that $\Sigma_\lambda = \{\xi\colon A^0(\xi)=\lambda\}$ is a *strongly convex surface* i.e.

$$\pm\sum_{j,k} A^0_{\xi_j\xi_k}(\xi)\eta_j\eta_k \ge \epsilon|\eta|^2 \qquad \forall\xi\colon A^0(\xi)=\lambda \ \ \forall\eta\colon \sum_j A^0_{\xi_j}(\xi)\eta_j = 0, \tag{16}$$

where the sign depends on the connected component of Σ_λ, containing ξ.

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Condition (A).

For each $\theta_1,\ldots,\theta_d \in \Theta$ either $\theta_1,\ldots,\theta_d$ are linearly independent over $\mathbb{R}$ or they linearly dependent over $\mathbb{Z}$.

Assume also that

Condition (B).

For any arbitrarily large L and for any sufficiently large real number ω there are a finite symmetric about 0 set $\Theta' := \Theta'_{(L,\omega)} \subset (\Theta \cap \mathrm{B}(0,\omega))$ (with $\mathrm{B}(\xi, r)$ the ball of the radius r and center ξ) and a “cut-off” coefficients $b'_\theta := b'_{\theta,(L,\omega)}$, such that

$$B' := B'_{(L,\omega)}(x,\xi) := \sum_{\theta\in\Theta'} b'_\theta(\xi) e^{i\langle\theta,x\rangle} \qquad (17)$$

satisfies

$$\|D_x^\alpha D_\xi^\beta \bigl(B - B'\bigr)\|_{\mathscr{L}^\infty} \le \omega^{-L}(|\xi|+1)^m \qquad \forall \alpha,\beta\colon |\alpha|\le L, |\beta|\le L. \quad (18)$$

Remark 1.

1. Then

$$|D_\xi^\beta b_\theta| = O(|\theta|^{-\infty}(|\xi|+1)^m) \qquad \text{as} \quad |\theta|\to\infty \tag{19}$$

and

$$|D_\xi^\beta (b_\theta - b'_\theta)| = O(\omega^{-\infty}(|\xi|+1)^m). \tag{20}$$

Indeed, one suffices to observe that $b_\theta(\xi) = \mathrm{M}(B(x,\xi)e^{-i\langle\theta,x\rangle})$ etc.

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Indeed, one suffices to observe that $b_\theta(\xi) = \mathrm{M}(B(x,\xi)e^{-i\langle\theta,x\rangle})$ etc.

2. On the other hand, under additional assumption

$$\#\{\theta\in\Theta,\ |\theta|\le\omega\} = O(\omega^p) \qquad \text{as} \quad \omega\to\infty \tag{21}$$

for some p, (19) implies Condition (B) with $\Theta'_{(L,\omega)} := \Theta\cap \mathrm{B}(0,\omega)$. However we will need $\Theta'_{(L,\omega)}$ in the next condition.

Remark 1 (Continued).

3. We need only to estimate the operator norm of the difference between $B(x, hD)$ and $B'(x, hD)$ (from $\mathscr{H}^m$ to $\mathscr{L}^2$); therefore for differential operators we can weaken (18).

Remark 1 (Continued).

3. We need only to estimate the operator norm of the difference between $B(x, hD)$ and $B'(x, hD)$ (from $\mathscr{H}^m$ to $\mathscr{L}^2$); therefore for differential operators we can weaken (18).
4. While Condition (B) is Condition B of [PS3], adopted to our case, Condition (A) and Conditions (C), (D) below are borrowed without any modifications (except changing notations).

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$$\Theta'_K := \sum_{1\le i\le K} \Theta'. \tag{22}$$

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We say that $\mathfrak{V}$ is a *quasi-lattice subspace* of dimension q, if $\mathfrak{V}$ is a linear span of q linear independent vectors $\theta_1,\ldots,\theta_q \in \Theta'_K \setminus 0$. Obviously, the zero space is a quasi-lattice subspace of dimension 0 and $\mathbb{R}^d$ is a quasi-lattice subspace of dimension d.

We denote by $\mathcal{V}_q$ the collection of all quasi-lattice subspaces of dimension q and also $\mathcal{V} := \bigcup_{q\ge 0}\mathcal{V}_q$.

Consider $\mathfrak{V},\mathfrak{U}\in\mathcal{V}$. We say that these subspaces are *strongly distinct*, if neither of them is a subspace of the other one. Next, let $\widehat{(\mathfrak{V},\mathfrak{U})}\in[0,\pi/2]$ be the angle between them, i.e. the angle between $\mathfrak{V}\ominus\mathfrak{W}$ and $\mathfrak{U}\ominus\mathfrak{W}$, $\mathfrak{W}=\mathfrak{U}\cap\mathfrak{V}$. This angle is positive iff $\mathfrak{V}$ and $\mathfrak{U}$ are strongly distinct.

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Condition (C).

For each fixed L and K the sets $\Theta'_{(L,\omega)}$ satisfying (17) and (18) can be chosen in such a way that for sufficiently large ω we have

$$s(\omega) = s(\Theta'_K) \coloneqq \inf_{\mathfrak{V}, \mathfrak{U} \in \mathcal{V}} \sin(\widehat{(\mathfrak{V}, \mathfrak{U})}) \geq \omega^{-1} \tag{23}$$

and

$$r(\omega) \coloneqq \inf_{\theta \in \Theta'_K \setminus 0} |\theta| \geq \omega^{-1}, \tag{24}$$

where the implied constant (how large should ω be) depends on L and K.

Let $\mathfrak{V}$ be the span of $\theta_1, \ldots, \theta_q \in \Theta'_K \setminus 0$. Then due to Condition (A) each element of the set $\Theta'_K \cap \mathfrak{V}$ is a linear combination of $\theta_1, \ldots, \theta_q$ with rational coefficients. Since the set $\Theta'_K \cap \mathfrak{V}$ is finite, this implies that the set $\Theta'_\infty \cap \mathfrak{V}$ is discrete and is, therefore, a lattice in $\mathfrak{V}$. We denote this lattice by $\Gamma(\omega; \mathfrak{V})$.

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Our final condition states that this lattice cannot be too dense.

Condition (D).

We can choose $\Theta'_{(L;\omega)}$, satisfying Conditions (B) and (C) in such a way that for sufficiently large ω and for each $\mathfrak{V} \in \mathcal{V}$, $\mathfrak{V} \neq \mathbb{R}^d$, we have

$$\mathrm{vol}(\mathfrak{V}/\Gamma(\omega; \mathfrak{V})) \geq \omega^{-1}. \tag{25}$$

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2. Further, if Θ is a finite set and Condition (A) is fulfilled, then $\Theta_\infty := \bigcup_{K\ge 1}\Theta_K$ is a lattice and Conditions (B)–(D) are fulfilled.
3. Furthermore, the same is true, if Θ is an arithmetic sum of a finite set and a lattice.

Theorem 3 (Main Theorem).

Let A be a self-adjoint operator (13), *where A^0 satisfies* (1), (2), (4) *and* (16) *and B satisfies* (1).

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Let Conditions (A)–(D) be fulfilled. Then for $|\tau - \lambda| < \epsilon$, $\varepsilon \le h^{\vartheta}$, $\vartheta > 0$

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Corollary 4.

In the framework of Theorem 3

$$\mathsf{N}_{h,\varepsilon}(\tau) \sim \sum_{n\ge 0} \bar{\kappa}_n(\tau;\varepsilon) h^{-d+n}. \tag{27}$$

Proof (sketched)

Remark 5.

1. It follows from Section 4 of [Ivr1] that the contribution of the zone $\{\xi\colon |A^0(\xi)-\tau|\geq C_0\varepsilon+h^{1-\varsigma}\}$ to the remainder is negligible. Here and below $\varsigma>0$ is an arbitrarily small exponent.
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Therefore we restrict ourself by the analysis in the zone Ω_τ.
2. To upgrade (8) with $T = T_*$ (a small constant) to (8) with $T = T^*$ it is sufficient to prove that

$$|F_{t\to h^{-1}\tau}\bigl(\chi_T(t)\bigl(Q_{2x}u_h(x,y,t)\,{}^tQ_{1y}\bigr)\bigr|_{y=x}\bigr)| \le C_s h^{-d+s}, \qquad (28)$$

for $|\tau-\lambda|\le\epsilon$, $T\in[T_*,\ T^*]$ and $\chi\in\mathscr{C}_0^\infty([-1,-\frac{1}{2}]\cup[\frac{1}{2},1])$, where s is an arbitrarily large exponent.

Remark 6.

1. It suffices to prove asymptotics

$$e_h(x,x,\tau) = \sum_{0\le n\le M} \kappa_n(x,\tau) h^{-d+n} + O(h^{-d+M}) \tag{29}$$

with arbitrarily large fixed M. To do so we will use the *semiclassical Schrödinger operator method* with maximal time $T^* = h^{-M}$.

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3. First such replacement will be $B' := B'_{(L,\omega)}$ from Condition (B) with $\omega = h^{-\sigma}$, arbitrarily small $\sigma > 0$ and $L = 3M/\sigma$.
   So, from now Θ and B are effectively replaced by $\Theta' := \Theta'_{(L,\omega)}$ and $B'_{(L,\omega)}$ correspondingly.

Consider now the "gauge" transformation $A \mapsto e^{-i\varepsilon h^{-1}P} A e^{i\varepsilon h^{-1}P}$ with h-pseudodifferential operator P. Observe that

$$e^{-i\varepsilon h^{-1}P} A e^{i\varepsilon h^{-1}P} = A - i\varepsilon h^{-1}[P, A] + \sum_{2\le n\le K-1} \frac{1}{n!}(-i\varepsilon h^{-1})^n \operatorname{Ad}_P^n(A)$$

$$+ \int_0^1 \frac{1}{(K-1)!}(1-s)^{K-1}(-i\varepsilon h^{-1})^K e^{-i\varepsilon h^{-1}sP} \operatorname{Ad}_P^K(A) e^{i\varepsilon h^{-1}sP}\, ds, \qquad (30)$$

where $\operatorname{Ad}_P^0(A) = A$ and $\operatorname{Ad}_P^{n+1}(A) = [P, \operatorname{Ad}_P^n(A)]$ for $n = 0, 1, \ldots$.

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Thus *formally* we can compensate εB, taking

$$P = \sum_\theta ih\bigl(A^0(\xi + \theta h/2) - A^0(\xi - \theta h/2)\bigr)^{-1} b_\theta(\xi) e^{i\langle\theta,x\rangle}, \qquad (31)$$

so that

$$ih^{-1}[P, A^0] = B \implies ih^{-1}[P,A] = B + i\varepsilon h^{-1}[P,B]. \qquad (32)$$

Then perturbation εB is replaced by $\varepsilon^2 B'$, which is the right hand expression in (30) minus A^0, i.e.

$$B' = -ih^{-1}[P, B] + \sum_{2\le n\le K-1} \frac{1}{n!}\varepsilon^{n-2}(-ih^{-1})^n \operatorname{Ad}_P^n(A), \tag{33}$$

where we ignored the remainder.

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New perturbation, again formally, has a magnitude of ε^2. Repeating this process we will make a perturbation negligible.

Remark 7.

However, we need to address the following issues issues:

1. Denominator $h^{-1}\bigl(A^0(\xi+\theta h/2)-A^0(\xi-\theta h/2)\bigr)=\langle \nabla_\xi A^0,\theta\rangle + O(h^{1-\sigma})$ could be small.

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2. In B' set Θ' increases: $\varepsilon^2 B'=\varepsilon^2 B'_2+\varepsilon^3 B'_3+\ldots+\varepsilon^M B'_M$, where for B'_j the frequency set is Θ'_j (the arithmetic sum of j copies of Θ').

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3. We need to prove that the remainder is negligible.
4. This transformation was used in Section 9 of [PS3] (etc); in contrast to these papers we use Weyl quantization instead of pq-quantization, and have therefore $\bigl(A^0(\xi+\theta h/2)-A^0(\xi-\theta h/2)\bigr)$ instead of $\bigl(A^0(\xi+\theta h)-A^0(\xi)\bigr)$.

One can see easily that if inequality

$$|\langle \nabla_\xi A^0(\xi), \theta\rangle| \ge \gamma := \varepsilon^{\frac{1}{2}} h^{-\delta} \tag{34}$$

holds for all $\theta \in \Theta'_K$, then the terms could be estimated by $h^{\delta n}$ and our construction works with $K = 3M/\delta$. Here and below without any loss of the generality we assume that $\varepsilon \ge h$; so, in fact, $h^\vartheta \ge \varepsilon \ge h$.

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Indeed, if $P = P(x, hD)$ has the symbol, satisfying

$$|D_\xi^\alpha D_x^\beta P| \le C_{\alpha\beta}\gamma^{-1-|\alpha|} \qquad \forall \alpha, \beta, \tag{35}$$

then $B' = \varepsilon h^{-1}[P, B]$ has a symbol, satisfying

$$|D_\xi^\alpha D_x^\beta B'| \le c'_{\alpha\beta}\varepsilon\gamma^{-2-|\alpha|} \qquad \forall \alpha, \beta, \tag{36}$$

so indeed $\varepsilon' = \varepsilon h^{-1}\gamma^{-2}$.

Then we can eliminate a perturbation completely, save terms with the frequency 0, both old and new. The set of ξ satisfying (34) for all $\theta \in \Theta'_K$ we call *non-resonant zone* and denote by $\mathcal{Z}$. Thus, we arrive to

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Proposition 8.

Let $Q = Q(hD)$ with the symbol supported in $\mathcal{Z} \cap \Omega$ and satisfying

$$|D^\alpha Q_j| \le C_\alpha h^{-(1-\varsigma)|\alpha|} \qquad \forall \alpha. \tag{37}$$

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Then there exists a pseudo-differential operator $P = P(x, hD)$ with the symbol, satisfying (37) and such that

$$\bigl(e^{-i\varepsilon h^{-1}P} A e^{i\varepsilon h^{-1}P} - A''\bigr) Q \equiv 0 \tag{38}$$

with

$$A'' = A^0(hD) + \varepsilon B''_0(hD) \tag{39}$$

modulo operator from $\mathscr{H}^m$ to $\mathscr{L}^2$ with the operator norm $O(h^{3M})$.

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1. This proposition is similar to Lemma 9.3 of [PS3]. However, in contrast to [PS1, PS2, PS3, MPS], after it is proven we do not write asymptotic decomposition there, but simply prove that singularities do not propagate with respect to ξ there.

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2. It is our second replacement of operator A; recall that the first one was based on Condition (B), and now we ignore the remainder after transformation, which is justified by Remark 6.

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Let $Q_j = Q_j(hD)$ with the symbols, satisfying (37) and let symbol of Q_1 be supported in $\mathcal{Z} \cap \Omega$.

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Let $Q_j = Q_j(hD)$ *with the symbols, satisfying* (37) *and let symbol of* Q_1 *be supported in* $\mathcal{Z} \cap \Omega$*. Let* $\operatorname{dist}(\operatorname{supp}(Q_1), \operatorname{supp}(Q_2)) \ge c\gamma$*. Then*

$$\| Q_2 e^{ih^{-1}tA} Q_1 \| = O(h^{2M}) \qquad \text{as} \quad |t| \le T^* = h^{-M}. \tag{40}$$

Consider now *resonant zone*

$$\Lambda := \bigcup_{\theta\in\Theta'_K\setminus 0} \Lambda(\theta), \tag{41}$$

where $\Lambda(\theta)$ is the set of ξ, violating (34) for given θ:

$$\Lambda(\theta) = \Lambda_\delta(\theta) := \{\xi\colon |\langle\nabla_\xi A^0(\xi),\theta\rangle| \ge \gamma = c\varepsilon^{\frac{1}{2}}h^{-\delta}\}. \tag{42}$$

Consider the easiest case $d=2$ (in the trivial case $d=1$ there is no resonant zone). Due to assumption (16) for each θ

$$\mathsf{mes}_1(\Lambda(\theta)\cap\Sigma_\lambda)\le C\gamma. \tag{43}$$

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Note, $\#\Theta'_K \le Ch^{-\sigma}$ due to Condition (C). Thus $\mathrm{mes}_1(\Lambda \cap \Sigma_\lambda) \le \gamma h^{-\sigma}$. Recall, that $\sigma > 0$ is arbitrarily small.

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Thus, $\nabla_\xi A^0(\xi)$ does not change by more than $\gamma h^{-\sigma}$ and since σ ais arbitrarily small we conclude that (40) also holds for Q_1, supported in the resonant zone. Therefore

Proposition 11.

Estimate (40) *holds for all* Q_1*,* Q_2 *satisfying* (37) *and*

$$\mathrm{dist}(\mathrm{supp}(Q_1), \mathrm{supp}(Q_2)) \ge \gamma. \tag{44}$$

Consider case $d \geq 2$. Due Conditions (A), (C) and (D) we can cover $\Lambda \cap \Omega_\tau$ by Λ^*,

$$\Lambda \cap \Omega_\tau \subset \Lambda^* = \bigcup_{1 \leq j \leq d-1} \Lambda_j^*, \tag{45}$$

defined as

Let $\xi \in \Omega_\tau$; then $\xi \in \Lambda_j^*$ iff there exist $\theta_1, \ldots, \theta_j \in \Theta'_K$ which are linearly independent and such that $\xi \in \Lambda_{\delta_j}(\theta_k)$ for all $k = 1, \ldots, j$,

where $0 < \delta = \delta_1 < \delta_2 < \ldots < \delta_{d-1}$ are arbitrarily fixed and we chose sufficiently small $\sigma > 0$ afterwards.

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where $0 < \delta = \delta_1 < \delta_2 < \dots < \delta_{d-1}$ are arbitrarily fixed and we chose sufficiently small $\sigma > 0$ afterwards.

Further, due to Conditions (A), (C), (D) and (16) $\Lambda_{d-1}^* \cap \Omega_\tau$ could be covered by no more than γ_{d-1}-vicinities of some points ξ_ι, $\iota = 1, \dots, \omega^g$, $g = g(d)$. Recall that $\Omega_\tau := \{\xi \colon |A^0(\xi) - \tau| \leq C_0\varepsilon + h^{1-\varsigma}\}$.

Consider some connected component Ξ of Λ^*_j. Let some point $\bar{\xi}$ of it belong to $\bigcap_{1\le k\le j}\Lambda_{\delta_j}(\theta_k)\cap\Omega_\tau$ with linearly independent $\theta_1,\dots,\theta_j$. Observe that $\operatorname{diam}(\bigcap_{1\le k\le j}\Lambda_{\delta_j}(\theta_k)\cap\Omega)\le c\gamma_j$ due to strong convexity assumption (16). Then this set either intersects or does not intersect with $\Lambda^*_{j+1}\cap\Omega$. In the former case we include it to Λ^*_{j+1} and exclude it from Λ^*_j.

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After we redefined Λ^*_j we arrive to the following proposition:

Proposition 12.

Equation (45) still holds where now each connected component Ξ of Λ^*_j has the following properties:

1. $\operatorname{diam}\Xi\le c\gamma_j$.
2. *There exist linearly independent $\theta_1,\ldots,\theta_j\in\Theta'_K$, such that for each $\xi\in\Xi$ $|\langle\nabla_\xi A^0(\xi),\theta\rangle|\le c_j\gamma_j$ for all $\theta\in\mathfrak{V}\cap(\Theta'_K\setminus 0)$ and $|\langle\nabla_\xi A^0(\xi),\theta\rangle|\ge\epsilon_j\gamma_{j+1}$ for all $\theta\in\Theta'_K\setminus\mathfrak{V})$ with $\mathfrak{V}=\operatorname{span}(\theta_1,\ldots,\theta_j)$.*

Now we generalize Proposition 8:

Proposition 13.

Let $Q = Q(hD)$ with the symbol supported in the connected component Ξ of Λ_j^*, corresponding to subspace $\mathfrak{V}$, and satisfying (37)*. Then there exists a pseudo-differential operator $P = P(x, hD)$ with the symbol, satisfying* (35) *and such that*

$$\bigl(e^{-i\varepsilon h^{-1}P} A e^{i\varepsilon h^{-1}P} - A''\bigr) Q \equiv 0 \tag{46}$$

modulo operator from $\mathscr{H}^m$ to $\mathscr{L}^2$ with the operator norm $O(h^{3M})$, where $A'' = A^0 + \varepsilon B''(x, hD)$, where B'' is an operator with Weyl symbol

$$B''(x,\xi) = \sum_{\theta \in \Theta'_K \cap \mathfrak{V}} b_{\mathfrak{V},\theta}(\xi) e^{i\langle \theta, x\rangle}. \tag{47}$$

Then we arrive to

Proposition 14.

Let $Q_j = Q_j(hD)$ *with the symbols, satisfying* (37) *and let symbol of* Q_1 *be supported in* Λ_j^* *.*
Let $\operatorname{dist}(\operatorname{supp}(Q_1), \operatorname{supp}(Q_2)) \geq C_0\gamma_j$*. Then* $\|Q_2 e^{ih^{-1}tA} Q_1\| = O(h^{2M})$ *for* $|t| \leq T_* = h^{-M}$*.*

Then we arrive to

Proposition 14.

Let $Q_j = Q_j(hD)$ with the symbols, satisfying (37) and let symbol of Q_1 be supported in Λ_j^* .
Let $\operatorname{dist}(\operatorname{supp}(Q_1), \operatorname{supp}(Q_2)) \ge C_0\gamma_j$. Then $\|Q_2 e^{ih^{-1}tA} Q_1\| = O(h^{2M})$ for $|t| \le T_* = h^{-M}$.

Next we arrive to the following proposition:

Proposition 15.

Let Q_1, Q_2 satisfy (37) and $\operatorname{supp}(Q_1) \subset \Omega$. Then for $T_* \le T \le T^*$

$$F_{t\to h^{-1}\tau}\bigl(\chi_T(t) Q_{2x} u(x,y,t)\, {}^tQ_{1y}\bigr) = O(h^{2M}). \tag{48}$$

Now we conclude that

$$F_{t\to h^{-1}\tau}\Bigl([\bar{\chi}_T(t)-\bar{\chi}_{T_*}(t)]Q_{2x}u(x,y,t)\,{}^tQ_{1y}\Bigr)\Bigr|_{x=y}=O(h^{2M}) \tag{49}$$

Now we conclude that

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and since

$$F_{t\to h^{-1}\tau}\bigl(\bar\chi_T(t)Q_{2x}u(x,y,t)\,{}^tQ_{1y}\bigr)\bigr|_{x=y}=\sum_{0\le n\le M}\kappa'_n(x,\varepsilon)h^{1-d+n}+O(h^{M+1}) \tag{50}$$

holds for $T=T_*$, it also holds for $T=T^*$.

Now we conclude that

$$F_{t\to h^{-1}\tau}\bigl([\bar\chi_T(t)-\bar\chi_{T_*}(t)]Q_{2x}u(x,y,t)\,{}^tQ_{1y}\bigr)\bigr|_{x=y}=O(h^{2M}) \tag{49}$$

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Finally, Hörmander's Tauberian theorem implies Theorem 3.

Generalizations and Discussion

Remark 16.

1. One can generalize Theorem 3 to elliptic matrix operators, assuming that eigenvalues of $A^0(\xi)$ are simple and satisfy assumptions of this theorem.

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Remark 16.

1. One can generalize Theorem 3 to elliptic matrix operators, assuming that eigenvalues of $A^0(\xi)$ are simple and satisfy assumptions of this theorem.
2. As $d = 2$ one can replace strong convexity condition (16) by much weaker nondegeneracy assumption.

Remark 17.

1. One can generalize Theorem 3 to operators

$$A = A^0(hD) + \varepsilon V(x, HD), \tag{51}$$

where

$$|D_\xi^\alpha D_x^\beta V(x,\xi)| \le c_{\alpha\beta}(|\xi|+1)^m(|x|+1)^{-\delta-|\beta|} \qquad \forall \alpha,\beta\ \forall x,\xi \tag{52}$$

provided $\varepsilon \le \epsilon_0$.

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provided $\varepsilon \le \epsilon_0$.

2. One can generalize Theorem 3 to operators

$$A = A^0(hD) + \varepsilon\bigl(B(x,hD) + V(x,hD)\bigr), \tag{53}$$

where $B(x, hD)$ satisfies conditions of Theorem 3 and V satisfies (52), and even for more general operators.

Remark 18.

1. It also follows from Corollary 4 that

$$\frac{1}{\nu}\Big[\mathsf{N}_{h,\varepsilon}(\tau+\nu)-\mathsf{N}_{h,\varepsilon}(\tau)\Big]=\frac{1}{\nu}\Big[\mathcal{N}_{h,\varepsilon}(\tau+\nu)-\mathcal{N}_{h,\varepsilon}(\tau)\Big]+O(h^{\infty}) \quad (54)$$

provided $\nu\ge h^M$, where $\mathcal{N}_{h,\varepsilon}(\tau)$ is the right-hand expression of (27).

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provided $\nu \ge h^M$, where $\mathcal{N}_{h,\varepsilon}(\tau)$ is the right-hand expression of (27).

2. The question remains, if (54) holds for smaller ν, in particular, if it holds in $\nu\to 0$ limit? If the latter holds, then

$$\frac{\partial}{\partial\tau}\mathsf{N}_{h,\varepsilon}(\tau)=\frac{\partial}{\partial\tau}\mathcal{N}_{h,\varepsilon}(\tau)+O(h^{\infty}) \tag{55}$$

and we call the left-hand expression the *density of states*.

Remark 18 (Continued).

3. It definitely is not necessarily true, at least in dimension 1. From now on we consider only asymptotics with respect to $\tau \to +\infty$. Let $A = \Delta + V(x)$ with periodic V. It is well-known that for $d = 1$ and generic periodic V all spectral gaps are open which contradicts to

$$\frac{\partial}{\partial \tau}\mathsf{N}(\tau) = \frac{\partial}{\partial \tau}\mathcal{N}(\tau) + O(\tau^{-\infty}). \tag{56}$$

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4. On the other hand, this objection does not work in case $d\geq 2$ since only several the lowest spectral gaps are open (Bethe-Sommerfeld conjecture, proven in [PS]).

Remark 18 (End).

5. Further, one can differentiate $e(x, x, \tau^2)$ if $d \geq 2$ and V is compactly supported.

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6. Moreover, we can differentiate complete asymptotics of the *Birman-Schwinger spectral shift function*

$$\xi(\tau) := \int \bigl(e(x,x,\tau^2) - e^0(x,x,\tau^2)\bigr)\,dx \sim \sum_{n\geq 0} \bar{\kappa}_n \tau^{-d+n}, \tag{57}$$

with

$$\bar{\kappa}_n := \int (\kappa_n(x) - \kappa_n^0)\,dx, \tag{58}$$

where $e^0(x,y,\tau)$ and κ_n^0 correspond to $A^0 = \Delta$.

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where $e^0(x,y,\tau)$ and κ_n^0 correspond to $A^0=\Delta$. In the case of $A=\Delta$ in the exterior of smooth, compact and non-trapping obstacle and $A^0=\Delta$ in $\mathbb{R}^d$ such asymptotics was derived in [PP].

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