Eigenvalue asymptotics for Steklov's problem in the domain with edges

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Introduction

Let X be a compact connected $(d+1)$-dimensional Riemannian manifold with the boundary Y, regular enough to properly define operators J and Λ (manifolds with edges are of this type):

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For v, which is a restriction to Y of $\mathscr{C}^2$ function, we define $Jv = w$, where $\Delta w = 0$ in X, $w|_Y = v$, and $\Lambda v = -\partial_\nu Jv|_Y$, where Δ is a positive Laplacian (understood in the sense of forms, if needed), ν is a unit inner normal to Y.

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Proposition 1

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2. *It has a discrete accumulating to infinity spectrum with eigenvalues $0 = \lambda_0 < \lambda_1 \le \ldots$ could be obtained recurrently from the following variational problem:*

$$\int_X |\nabla w|^2\,dx \mapsto \min(=\lambda_n)$$
$$\text{as}\quad \int_Y |w|^2\,dx' = 1, \qquad \int_Y ww_k^\dagger\,dx' = 0 \quad \text{for}\ \ k = 0,\ldots,n-1. \tag{1}$$

Corollary 2

The number $\mathsf{N}(\lambda)$ *of eigenvalues of* Λ*, which are less than* λ*, equals to the maximal dimension of the linear space of* $\mathscr{C}^\infty$*-functions, on which the quadratic form*

$$\int_X |\nabla w|^2\,dx - \lambda \int_Y |w|^2\,dx' \tag{2}$$

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Our purpose is to derive asymptotics on $\mathsf{N}(\lambda)$ as $\lambda\to +\infty$.

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$$\mathsf{N}(\lambda) = \kappa_0\lambda^d + O(\lambda^{d-1}) \qquad \text{as} \ \ \lambda \to \infty \tag{3}$$

with the standard coefficient $\kappa_0 = (2\pi)^{-d}\omega_d \operatorname{mes}(Y)$, where $\operatorname{mes}(Y)$ means d-dimensional volume of Y, ω_d is the volume of the unit ball in $\mathbb{R}^d$.

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Moreover, if the set of all periodic geodesics of Y has measure 0, then

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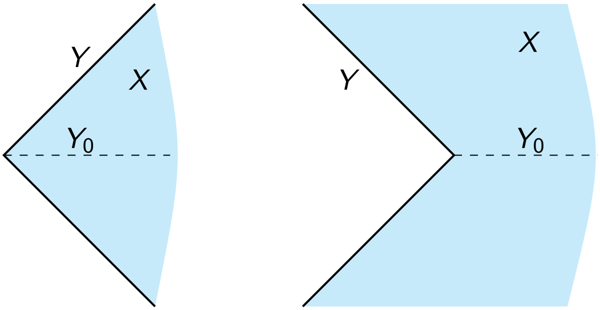
Remark 1

In this case one can easily calculate κ_1.

Consider now the case when Y has edges, i.e. each point $y \in Y$ has a neighbourhood U in X, which is the diffeomorphic either to $\mathbb{R}^+ \times \mathbb{R}^d$ (a regular point), or to $\mathbb{R}^{+\,2} \times \mathbb{R}^{d-1}$ (an inner edge point), or to $(\mathbb{R}^2 \setminus \mathbb{R}^{-\,2}) \times \mathbb{R}^{d-1}$ (an outer edge point).



(a) $\alpha \in (0, \pi)$ (b) $\alpha \in (\pi, 2\pi)$

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Our goal

We are looking for two-term asymptotics like (4) or (32) below.

Dirichlet-to-Neumann operator

Toy-model: dihedral angle

Let $Z = \mathbb{R}^{d-1}$ with Euclidean metrics, $X = \mathcal{X} \times Z$, $Y = \mathcal{Y} \times Z$ where $\mathcal{X}$ is a planar angle of solution α, $0 < \alpha \le 2\pi$, $\mathcal{Y} = \mathcal{Y}_1 \cup \mathcal{Y}_2$, $\mathcal{Y}_j$ are rays.

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Then one can identify Y with $\mathbb{R}^d$ with coordinates (s, z), where $z \in Z$ and

- $s = \operatorname{dist}(y, Z)$ for for a point $y \in Y_1 = \mathcal{Y}_1 \times Z$,
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Then we have a Euclidean metrics and a positive Laplacian Δ_Y on Y.

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1. We can consider any $\alpha > 0$, including $\alpha > 2\pi$ (but then we need to address some issues with the domain of operator).
2. If $\alpha = \pi$, then $\Lambda = \Delta_Y^{1/2}$.

For a toy-model we can make a partial Fourier transform $F_{z\to\zeta}$ and then study equation in the planar angle:

$$\Delta_2 w + w = 0, \tag{5}$$

where Δ_2 is a positive 2D-Laplacian and we made also a change of variables $x'' \mapsto |\zeta|\cdot x''$, $x''=(x_1,x_2)$.

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Denote by $\hat{J}$ and $\hat{\Lambda}$ operators J and Λ for (5).
Then we can use separation of variables. Singularities at the vertex are the same as for $\Delta_2 w = 0$, $w|_Y = 0$ and they are combinations of $r^{\pi n/\alpha} \sin(\pi n\theta/\alpha)$ with $n = 1, 2, \ldots$, where $(r, \theta) \in \mathbb{R}^+ \times (0, \alpha)$ are polar coordinates.

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This shows the role of α: if $\alpha\in(0,\pi)$ those functions are in $\mathscr{H}^{\sigma}_{\mathsf{loc}}(\mathcal{X})$ with $\sigma < 1+\pi n/\alpha$, and $\partial_\nu w|_Y$ belong to $\mathscr{H}^{\sigma-3/2}_{\mathsf{loc}}(\mathcal{Y})$.

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Definition 3

Due to our assumption either $\alpha(z) \in (0, \pi)$ or $\alpha(z) \in (\pi, 2\pi)$. Let Z_j be a connected component of Z.

- Z_j is an inner edge if $\alpha(z) \in (0, \pi)$ on Z_j, and
- Z_j is an outer edge if $\alpha(z) \in (\pi, 2\pi)$ on Z_j.

Proposition 4

The following are bounded operators

$$\Delta_{\mathrm{D}}^{-1} : \mathscr{H}^{\sigma}(X) \to \mathscr{H}^{\sigma+2}(X), \tag{6}$$

$$J : \mathscr{H}^{\sigma+\frac{3}{2}}(Y) \to \mathscr{H}^{\sigma+2}(X), \tag{7}$$

$$\Lambda : \mathscr{H}^{\sigma+\frac{3}{2}}(Y) \to \mathscr{H}^{\sigma+\frac{1}{2}}(X), \tag{8}$$

where Δ_{D} *is an operator* Δ *with zero Dirichlet boundary conditions on* Y *and*

- $\sigma \in [-\frac{1}{2}, 0]$, *if* $\alpha(z) \in (0,\pi)$ $\forall z \in Z$, *and*
- $\sigma \in [-\frac{1}{2}, \bar{\sigma})$ *with* $\bar{\sigma} = \pi/\bar{\alpha} - 1$, $\bar{\alpha} = \max_{z\in Z} \alpha(z)$ *otherwise.*

Proposition 5

1. *For equation* (5) *in* $\mathcal{X}$

$$\hat{\Lambda} - (D_s^2 + 1)^{1/2} = \sum_{j+k\le 1} D_s^j K_{jk} D_s^k\,, \tag{9}$$

where K_{jk} *have Schwartz kernels* $K_{jk}(s,s')$ *such that*

$$|D_s^p D_{s'}^q K_{jk}(s,t')| \le \\ C_{pq}|s|^{-(\bar\sigma-p)_-}|s'|^{-(\bar\sigma-q)_-}(|s|+|s'|)^{-p-q+(\bar\sigma-p)_-+(\bar\sigma-q)_-} \tag{10}$$

and $r_\pm := \max(\pm r, 0)$.

Proposition 5 continued

2. In the general case

$$\Lambda - \Delta_Y^{1/2} = b + \sum_{j+k\le 1} D_{x_1}^j K_{jk} D_{y_1}^k\,, \tag{11}$$

where b a bounded operator, and K_{jk} have Schwartz kernels such that

$$\begin{aligned}|D_x^\alpha D_y^\beta K_{jk}(x,y)| \le C_{\alpha\beta}|x_1|^{-(\bar\sigma-\alpha_1)_-}|y_1|^{-(\bar\sigma-\beta_1)_-}\times\\ \bigl(|x_1|+|y_1|+|x'-y'|\bigr)^{-d-1+j+k-|\alpha|-|\beta|+(\bar\sigma-\alpha_1)_-+(\bar\sigma-\beta_1)_-}\end{aligned} \tag{12}$$

and Δ_Y is a positive Laplacian on Y. Obviously, metrics on Y is continuous and piecewise $\mathscr{C}^\infty$.

More about toy-model

Obviously

$$(\hat{\Lambda} v, v)_{\mathcal{Y}} = \|\nabla w\|^2 + \|w\|^2, \tag{13}$$

where

$$\Delta_2 w + w = 0 \quad \text{in} \quad \mathcal{X}, \qquad w|_{\mathcal{Y}} = v \tag{14}$$

and we consider norms and inner products in $\mathscr{L}^2(\mathcal{X})$ or $\mathscr{L}^2(\mathcal{Y})$.

Let $\mathcal{Y}_0$ be a bisector of $\mathcal{X}$; we can assume that $\mathcal{Y}_0 = \{(x,y)\colon y=0, x>0\}$. Then if v is symmetric or antisymmetric with respect to y, so are $w=\hat{J}v$ and $\hat{\Lambda}v$.

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Multiplying (5) by $(\nu_x w_x + \nu_y w_y)$ and by $(yw_x - xw_y)$ and integrating over half-angle, one can prove easily

$$\int_{\mathcal{Y}_k} \bigl(|\hat{\Lambda}v|^2 - |v_s|^2 - |v|^2\bigr)\, ds = \begin{cases} -\cos(\alpha/2)\displaystyle\int_{\mathcal{Y}_0} \bigl(|w_x|^2 + |w|^2\bigr)\, dx \\ \cos(\alpha/2)\displaystyle\int_{\mathcal{Y}_0} |w_y|^2\, dx \end{cases} \tag{15}$$

and

$$\int_{\mathcal{Y}_k} \bigl(|\hat{\Lambda}v|^2 - |v_s|^2 - |v|^2\bigr)\, |s| ds = \begin{cases} \displaystyle\int_{\mathcal{Y}_0} \bigl(|w_x|^2 + |w|^2\bigr)\, x dx \\ -\displaystyle\int_{\mathcal{Y}_0} |w_y|^2\, x dx \end{cases} \tag{16}$$

for symmetric and antisymmetric v respectively.

Further, one can prove

$$\partial_\alpha(\hat{\Lambda}(\alpha)v, v)_{\mathcal{Y}} = -\int_{\mathcal{Y}_k} \bigl(|\hat{\Lambda}v|^2 - |\partial_s v|^2 - |v|^2\bigr)\,|s|\,ds, \qquad k = 1, 2. \tag{17}$$

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Combining (15)–(17) we arrive to

Proposition 6

1. *On symmetric functions:* $\hat{\Lambda}(\alpha)$ *is monotone increasing function of* α. *Also,* $\hat{\Lambda}(\alpha)^2 - (D_s^2 + I)$ *is negative definite for* $\alpha \in (0, \pi)$ *and positive definite for* $\alpha \in (\pi, 2\pi)$.

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2. *On antisymmetric functions: $\hat{\Lambda}(\alpha)$ is monotone decreasing function of α. Also, $\hat{\Lambda}(\alpha)^2 - (D_s^2 + I)$ is positive definite for $\alpha \in (0, \pi)$ and negative definite for $\alpha \in (\pi, 2\pi)$.*

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Problem 1

Prove that $[1, \infty)$ is always absolutely continuous spectrum; in particular, that there are no embedded eigenvalues.

Remark 4

Paper [KP] deals with the discrete eigenvalues of Δ_2 in the planar sector under Robin boundary condition $(\partial_\nu + \gamma)w|_Y = 0$, $\gamma > 0$.Then eigenvalues τ of Λ and eigenvalues μ of that problem are related through Birman-Schwinger principle and scaling: $\mu_k = -\tau_k^{-2}\gamma^2$. Some of the results:

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2. Theorem 2.3 states that for $\alpha \in (0, \pi)$ the bottom eigenvalue $-\gamma^2/\sin^2(\alpha/2)$ is simple and the corresponding eigenfunction is $\exp(-\gamma x_1/\sin(\alpha/2)$.
3. Theorem 3.6 states that for $\alpha \in [\frac{\pi}{3}, \pi)$ there is no other eigenvalues in $(0, 1)$, while Theorem 4.1 implies that the number of such eigenvalues is $\asymp \alpha^{-2}$ as $\alpha \to 0$ [a].

[a] In fact, the compete asymptotic expansion of the eigenvalues is derived in Theorem 4.16 of [KP].

Propagation of singularities. I

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Locally we assume that X is a dihedral angle, $X = \mathcal{X} \times Z$, $Y = \mathcal{Y} \times Z$ like in the toy-model, albeit metrics is not Euclidean.

We need the following auxillary (easy to prove)

Proposition 8

1. *Let* $q_j = q_j(\xi')$ $(j = 1, 2)$ *be two symbols, constant as* $|\xi'| \ge C$. *Assume that the* $\operatorname{dist}(\operatorname{supp}(q_1), \operatorname{supp}(q_2)) \ge \epsilon$. *Consider* h*-pseudodifferential operators* $Q_j = q_j^{\mathrm{w}}(h^{-1}D')$, $j = 1, 2$. *Then the operator norms of*

$$Q_1 \Delta_{\mathrm{D}}^{-1} Q_2 : \mathscr{H}^{\sigma}(X) \to \mathscr{H}^{\sigma+2}(X), \tag{18}$$

$$Q_1 J Q_2 : \mathscr{H}^{\sigma+\frac{3}{2}}(Y) \to \mathscr{H}^{\sigma+2}(X), \tag{19}$$

$$Q_1 S Q_2 : \mathscr{H}^{\sigma+\frac{3}{2}}(Y) \to \mathscr{H}^{\sigma+\frac{1}{2}}(X) \tag{20}$$

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   do not exceed $C'h^m$ *with arbitrarily large* m, *for* σ *defined in Proposition 4.*

2. *Let* $Q_j(x')$ $(j = 1, 2)$ *be two functions. Then all three operators in are infinitely smoothing by* x'.

Let u be Schwartz kernel of $e^{ih^{-1}tA}$.

Proposition 9

Consider h-pseudodifferential operator $Q = q^{\mathrm{w}}(x', h^{-1}D')$ *where* q *vanishes* $\{|\xi'| \le c_0\}$. *Let* $\chi \in \mathscr{C}_0^\infty(\mathbb{R})$, $T \ge h^{1-\delta}$. *Then operator norms of*

$$F_{t\to\tau}\chi_T(t)Q_x u \qquad \textit{and} \qquad F_{t\to\tau}\chi_T(t)u\,{}^tQ_y$$

do not exceed $C'_T h^m$ *for* $\tau \le c$ *and* $c_0 = c_0(c)$ *where here and below* m *is arbitrarily large.*

Let u be Schwartz kernel of $e^{ih^{-1}tA}$.

Proposition 9

Consider h-pseudodifferential operator $Q = q^{\mathrm{w}}(x', h^{-1}D')$ *where* q *vanishes* $\{|\xi'| \le c_0\}$*. Let* $\chi \in \mathscr{C}_0^\infty(\mathbb{R})$*,* $T \ge h^{1-\delta}$*. Then operator norms of*

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do not exceed $C'_T h^m$ *for* $\tau \le c$ *and* $c_0 = c_0(c)$ *where here and below* m *is arbitrarily large.*

One need to consider $v = e^{ih^{-1}tA}f$, $f \in \mathscr{H}^1(Y)$, $\|f\|_{\mathscr{L}^2(Y)} = 1$ and observe that it satisfies $(hD_t - A)v = 0$. One can prove that operator $(hD_t - A)$ is elliptic in $\{|\xi'| \ge c_0, \tau \le c\}$ with $c_0 = C_0(\alpha)c$ (despite being of unknown nature the notion of ellipticity is clear) and also local with respect to x'.

Therefore, in what follows

Remark 5

Studying energy levels $\tau \le c$ we can always apply cut-out domain $\{|\xi'| \ge c_0\}$.

Proposition 10

For h-pseudodifferential operator $Q = q^{\mathrm{w}}(x, hD')$ the following formula connecting commutators $[\Delta, Q]$ and $[\Lambda + \partial_\nu, Q]$ holds:

$$-\mathrm{Re}\, i([\Delta, Q]Jv, Jv)_X = \mathrm{Re}\, i(([\Lambda, Q] + [\partial_\nu, Q])v, v)_Y \tag{21}$$

where v denotes any function on Y and $w = Jv$ its continuation as a harmonic function.

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where v denotes any function on Y and $w = Jv$ its continuation as a harmonic function.

Indeed,

$$\begin{aligned}
0 = (Q\Delta w, w)_X = (\Delta Qw, w) - ([\Delta, Q]w, w)_X = \\
- ([\Delta, Q]w, w)_X + (Qw, \Delta w)_X - (\partial_\nu Qw, w)_Y + (Qw, \partial_\nu w)_Y = \\
- ([\Delta, Q]w, w)_X - (Q\partial_\nu w, w)_Y - ([\partial_\nu, Q]w, w)_Y + (Qw, \partial_\nu w)_Y = \\
- ([\Delta, Q]w, w)_X + (Q\Lambda v, v)_Y - ([\partial_\nu, Q]v, v)_Y - (v, Q\Lambda v)_Y = \\
- ([\Delta, Q]w, w)_X - (\Lambda Qv, v)_Y - ([\partial_\nu, Q]v, v)_Y,
\end{aligned}$$

which implies (21).

Now we can prove that at energy levels $\tau \le c$ the propagation speed with respects to x and ξ' do not exceed $C_0 = C_0(c)$.

Proposition 11

Let $Q_j = q_j^{\mathrm{w}}(x', hD')$ *and* $\operatorname{dist}(\operatorname{supp}(q_1), \operatorname{supp}(q_2)) \ge C_0 T$ *with fixed* $T > 0$. *Let* $\chi \in \mathscr{C}_0^\infty([-1,1])$. *Then for* $\tau \le c$

$$|F_{t\to h^{-1}\tau}\bigl(\chi_T(t) Q_{1x} u\, {}^t Q_{2y}\bigr)| \le Ch^m \tag{22}$$

with arbitrarily large m.

The proof is the standard for propagation with respect to x', ξ': we consider $\phi(x', \xi', t)$ and prove that under the microhyperbolicity condition

$$\phi_t - \{|\xi'|, \phi\} \geq \epsilon_0 \tag{23}$$

which is equivalent to

$$2\phi_t - |\xi'|^{-1}\{|\xi'|^2, \phi\} \geq 2\epsilon_0 \tag{24}$$

our standard propagation theorem (see Theorem 2.1.2 of [Ivr2]) holds, just repeating arguments of its proof, using (21) and the fact that

$$\|Jv\|_{\mathscr{H}^{s+1/2}(X)} \asymp \|v\|_{\mathscr{H}^s(Y)} \qquad s \in [1, \frac{1}{2}], v \in \mathfrak{D}(\Lambda). \tag{25}$$

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Then we plug $\phi(x', \xi', t) = \psi(x', \xi') - t$ with $|\nabla_{x',\xi'}\psi| \leq 1 - \epsilon_1$, and prove that (22) for $q_j = q_j(x', \xi')$.

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We need also prove that (22) holds for $q_j = q_j(x_1, x_2)$ but this is easy since for Λ is a first-order pseudodifferential operator in in $\{|x_1| + |x_2| \geq \epsilon\}$.

Next we prove that at energy levels $\tau = 1$ the propagation speed with respects to x' in the vicinity of $(0, \bar{\xi}')$ with $|\bar{\xi}'| \geq \epsilon_0$ is at least $\epsilon_1 = \epsilon_1(\epsilon_0)$.

Proposition 12

Let $Q_j = q_j^{\mathrm{w}}(x, hD')$ *and* $\operatorname{dist}_{x'}(\operatorname{supp}(q_1), \operatorname{supp}(q_2)) \leq \epsilon_1 T$ *with fixed* $T > 0$. *Let* $\chi \in \mathscr{C}_0^\infty([-1, -\frac{1}{2}] \cup [-\frac{1}{2}, -1])$. *Then for* $|\tau - 1| \leq \epsilon_0$ (22) *holds.*

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Proof.

After propagation theorem mentioned in the proof of Proposition 10 is proven we just plug $\phi(x', \xi'.t) = \psi(x', \xi') - \epsilon t$ with $\xi' \cdot \nabla_{x'} \psi \ge 1$. □

Corollary 13

In the framework of Proposition 12 consider $|\tau - 1| \le \epsilon$. *Then*

$$|F_{t\to h^{-1}\tau}\Gamma_x \chi_T(t) u\, {}^t Q_y| \le C h^{1-d+m} T^{-m} \tag{26}$$

and

$$|F_{t\to h^{-1}\tau}\Gamma_x \bigl(\bar\chi_{T'}(t) - \bar\chi_T(t)\bigr) u\, {}^t Q_y| \le C h^{1-d+m} T^{-m} \tag{27}$$

with arbitrarily large m, *provided* $\chi \in \mathscr{C}_0^\infty([-1,-\frac{1}{2}] \cup [\frac{1}{2}, 1])$, $\bar\chi \in \mathscr{C}_0^\infty([-1,1])$, $\bar\chi = 1$ *on* $[-\frac{1}{2},\frac{1}{2}]$, $h \le T \le T' \le T_0$ *with small constant* T_0.

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Proof.

For small constant T (26) follows directly from Proposition 12, after one proves easily that we can insert $D_{x'}$, $D_{y'}$ to the corresponding estimate, which is easy.

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with arbitrarily large m, provided $\chi \in \mathscr{C}_0^\infty([-1,-\frac{1}{2}]\cup[\frac{1}{2},1])$, $\bar\chi \in \mathscr{C}_0^\infty([-1,1])$, $\bar\chi = 1$ *on* $[-\frac{1}{2},\frac{1}{2}]$, $h \le T \le T' \le T_0$ *with small constant* T_0.

Proof.

For small constant T (26) follows directly from Proposition 12, after one proves easily that we can insert $D_{x'}$, $D_{y'}$ to the corresponding estimate, which is easy. For $h \le T \le T_0$ we use just rescaling like in [Ivr2]. Finally, (27) is obtained by the summation with respect to partition of unity with respect to t. □

Propagation of singularities. II

As far as asymptotics, not requiring an assumption of the global nature are concerned, we need only to consider zone $\{|\xi'| \le \epsilon\}$ and prove that on the energy level 0 singularities do not stick to Z.

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We will use arguments similar to those we used for fractional Laplacians in Section 8.5 of [Ivr2] and [Ivr1]. Let us take $Q = x_1 D_1 + x_2 D_2 - i/2$, which is also an operator, acting on Y and coinciding there $Q = sD_s - i/2$, which is self-adjoint in $\mathscr{L}^2(Y)$.

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For fractional Laplacians in $\{x\colon x_1>0\}$ we took $Q=x_1D_1-i/2$.

Observe that (21) holds for this choice of Q (the proof works).

Also observe that for Euclidean Laplacian $[\partial_\nu, Q] = -i\partial_\nu$. Then we arrive for Euclidean Laplacians to

$$\begin{multline}\mathrm{Re}\, i(([\Lambda, Q]v, v)_Y = 2\mathrm{Re}((D_1^2 + D_2^2)Jv, Jv)_X + \mathrm{Re}(\Lambda v, v)_Y = \\ \mathrm{Re}(\Lambda v, v)_Y - 2\mathrm{Re}(\Delta' Jv, Jv). \tag{28}\end{multline}$$

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To use the positive commutator method we need to ensure that

$$\operatorname{Re}(\Lambda v, v)_Y - 2\operatorname{Re}(\Delta' Jv, Jv) \geq \epsilon_0 \|v\|_Y^2 \tag{29}$$

and this is true also in the general case since at the given point metrics could be made Euclidean.

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However, for the toy-model (29) becomes

$$\operatorname{Re}(\hat{\Lambda} v, v)_{\mathcal{Y}} - 2\|\hat{J}v\|^2 \ge \epsilon_0 \|v\|_{\mathcal{Y}}^2 \tag{30}$$

on the energy level $\tau = |\xi'|^{-1}$.

Then, since

$$\|v\|^2_{\mathcal{Y}} \le \|\nabla w\| \cdot \|Jv\| \quad \text{and} \quad (\hat{\Lambda} v, v)_{\mathcal{Y}} = \|\nabla w\|^2 + \|w\|^2$$

with $w = \hat{J}v$, and $(\hat{\Lambda} v, v)_{\mathcal{Y}}$ is between $\tau(1 \mp \epsilon')\|v\|^2_{\mathcal{Y}}$ on the energy levels near τ, we conclude that (30) holds for $\tau \ge C_0$ and thus (29) holds for $|\xi'| \le C_0^{-1}$ on the energy levels near 1.

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Proposition 14

1. *Estimates* (26)–(27) *hold for* Q*, supported in* $\{|\xi'| \le \epsilon\}$*, and* $x\colon |x_1| \le \epsilon T$*,* $ch \le T \le T_0$*.*

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Then we conclude that in zone $\{|\xi'| \le \epsilon\}$ singularities do not stick to Z:

Proposition 14

1. *Estimates* (26)–(27) *hold for Q, supported in $\{|\xi'| \le \epsilon\}$, and $x\colon |x_1| \le \epsilon T$, $ch \le T \le T_0$.*
2. *Estimates* (26)–(27)*, with Γ_x replaced by Γ, hold for Q supported in $\{|\xi'| \le \epsilon\}$ and $ch \le T \le T_0$.*

Then, since

$$\|v\|^2_{\mathcal{Y}} \le \|\nabla w\| \cdot \|Jv\| \quad \text{and} \quad (\hat{\Lambda} v, v)_{\mathcal{Y}} = \|\nabla w\|^2 + \|w\|^2$$

with $w = \hat{J}v$, and $(\hat{\Lambda} v, v)_{\mathcal{Y}}$ is between $\tau(1 \mp \epsilon')\|v\|^2_{\mathcal{Y}}$ on the energy levels near τ, we conclude that (30) holds for $\tau \ge C_0$ and thus (29) holds for $|\xi'| \le C_0^{-1}$ on the energy levels near 1.

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2. *Estimates* (26)–(27), *with* Γ_x *replaced by* Γ, *hold for* Q *supported in* $\{|\xi'| \le \epsilon\}$ *and* $ch \le T \le T_0$.

So, estimates (26)–(27) with Γ_x replaced by Γ hold for $Q = I$.

It allows us to employ for $u(x;y;t)$ our standard successive approximations method with unperturbed operator which is a toy-model operator at point $(y',0)\in Z$.

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This concludes the proof of Theorem 15 below.

Main theorems

Theorem 15

1. *The following asymptotics hols*

$$\mathsf{N}(\lambda) = \kappa_0 \lambda^d + O(\lambda^{d-1}). \tag{31}$$

2. *Moreover, for* $0 < r \le 1$

$$\mathsf{N}(\lambda) * \lambda_+^{r-1} = \bigl(\kappa_0 \lambda^d + \kappa_1 \lambda^{d-1}\bigr) * \lambda_+^{r-1} + O(\lambda^{d-1}). \tag{32}$$

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1. First statement of Theorem 15 is rather easy, and could be proven also by the modified Seeley's method.

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Remark 6

1. First statement of Theorem 15 is rather easy, and could be proven also by the modified Seeley's method.
2. Further, asymptotics with the same remainder estimate (albeit with more terms), holds for $r > 1$.

Sharper asymptotics

Let us discuss sharper asymptotics and what are obstacles to derive them.

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$$\mathsf{N}(\lambda) = \kappa_0\lambda^d + \kappa_1\lambda^{d-1} + o(\lambda^{d-1}) \tag{33}$$

and similarly improved (32) (with the third term for $r = 1$):

$$\mathsf{N}(\lambda) * \lambda_+^{r-1} = \bigl(\kappa_0\lambda^d + \kappa_1\lambda^{d-1} + \kappa_2\lambda^{d-2}\bigr) * \lambda_+^{r-1} + o(\lambda^{d-1}), \tag{34}$$

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one needs a meaningful propagation theorem. Moreover, asymptotics with the same remainder estimate but with more terms holds for $r > 1$.

To prove such theorem one needs to prove that singularities do not stick to Z not only for $|\xi'| \le \epsilon\tau$ as we did, but for $|\xi'| \le (1-\epsilon)\tau$.

To prove this by the positive commutator method we need to prove that for a planar angle

$$\operatorname{Re}(\hat{\Lambda} v, v)_{\mathcal{Y}} \geq 2\|\hat{J} v\|^2, \tag{35}$$

which boils down to

$$\Delta_2 w + w = 0 \implies \|\nabla w\| \geq \|w\|\,. \tag{36}$$

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Proposition 16

1. *For* $\alpha \in (\pi, 2\pi)$ (35) *holds.*

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Proposition 16

1. *For* $\alpha \in (\pi, 2\pi)$ (35) *holds.*
2. *For* $\alpha \in (0, \pi)$ (35) *holds on antisymmetric functions only. (It fails on symmetric functions, f.e. on* $w = e^{-px}\cosh(qy)$ *with* $p^2 + q^2 = 1$, $0 < q < \cot(\alpha/2)p$*).*

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And, in contrast to the general case of "two medias", the angles would be the same.

However even if we prove that singularities do not stick to Z on such energy levels, we would need to consider branching billiards, as each trajectory hitting Z, reflects from it and also refracts.
And, in contrast to the general case of "two medias", the angles would be the same. This is a tough case since (as demonstrated by Yu. Safarov and D. Vassiliev, the set of the dead-end billiards could have a positive measure).

However even if we prove that singularities do not stick to Z on such energy levels, we would need to consider branching billiards, as each trajectory hitting Z, reflects from it and also refracts.
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Then (assuming that the edge Z is outer) asymptotics (33) and (34) hold.

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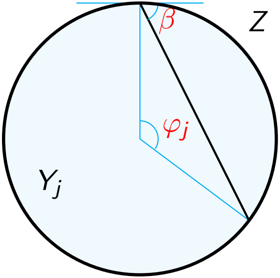

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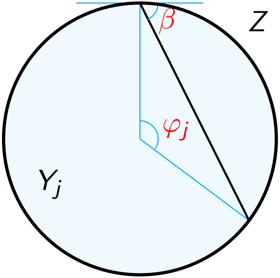

Assume that $\varphi_j(\beta)$ are analytic and $\varphi_j(\beta) \to 0$ as $\beta \to +0$. Then the measure of the set of partially periodic billiards is 0.

Miscellaneous

Remark 7

Let Y be smooth and $x_1 = \mathrm{dist}(x, Y)$.

1. Then $g(x,\xi) = g(x_1, x', \xi') + \xi_1^2$. Let $g_Y = g(0, x', \xi')$, $g'_Y = (\partial_\nu g)(0, x', \xi')$,

$$\kappa_1 = (2\pi)^{-d}\omega_d \int_{\{g_Y(x',\xi')=1\}} g'_Y(x',\xi')\, dx' d\xi' : dg_Y\,. \tag{37}$$

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2. In particular, for Euclidean $X \subset \mathbb{R}^{d+1}$

$$\kappa_1 = (2\pi)^{-d} d^2\omega_d^2 \int_Y \rho(y)\, dy\,, \tag{38}$$

where $\rho(y)$ is a mean curvature of Y as the surface in $\mathbb{R}^{d+1}$.

Remark 7 continued

3. In the domain with edges $\kappa_1 = \kappa_{1,Z\setminus Y} + \kappa_{1,Z}$ where $\kappa_{1,Z\setminus Y}$ is defined by (37) with Y replaced by $Y \setminus Z$ and

$$\kappa_{1,Z} = (2\pi)^{1-d}\omega_{d-1}\int_Z \varkappa(\alpha(y))\,dy\,, \tag{39}$$

$$\varkappa(\alpha) = \int_1^\infty \int \lambda^{-d}\Big(\mathbf{e}_\alpha(x_1,x_1,\lambda) - \pi^{-1}(\lambda-1)\Big)\,dx_1\,d\lambda\,, \tag{40}$$

$\mathrm{e}_\alpha(x_1,y_1,\lambda)$ is a Schwartz kernel of the spectral projector of $\hat{\Lambda}$ in the planar angle of solution α and $\pi^{-1}(\lambda-1)$ is a corresponding Weyl approximation.

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Remark 8

Our arguments hold not only for compact X but also for $X\subset\mathbb{R}^{d+1}$ with the compact complement and with the metrics. stabilizing to Euclidean at infinity.

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To do this we need to understand, how singularities propagate near inner edges. One can prove that there are plenty of singularities, concentrated in $Z \times \mathbb{R} \ni (x, t)$ and $\{|\xi'| < \tau\}$. This is similar to the Rayleigh waves. And, exactly like Rayleigh waves, those singularities do not prevent us from the sharper asympotics.

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What we need to prove is that the singularities in $\{|\xi'| < \tau\}$, coming from $Y \setminus Z$ transversally to Z, reflect and refract but leave Z instantly. In other words, that these two kinds of waves are completely separate. It is what I am trying to prove now.

To do this we need first

Problem 2

For a planar angle of solution $\alpha \in (0, \pi)$, let $\mathrm{n}(\alpha)$ be a number of eigenvalues of $\hat{\Lambda}(\alpha)$ in $(0, 1)$. Let $\mathcal{K}$ be the linear span of the corresponding eigenfunctions. Prove that (36) $\|\nabla w\| \geq \|w\|$ holds for $w = \hat{J}v$ with $v \in \mathcal{K}^{\perp}$. One can prove easily that $\|\nabla w\| = \|w\|$ for $w = \hat{J}v$ and eigenfunction v.

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