Etudes in Spectral Theory

Math Physics Seminar at Tel-Aviv University

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1. Distribution of Eigenvalues
   - Variational Methods
   - Tauberian Methods

2. Equidistribution of eigenfunctions

3. Can one hear the shape of the drum?

4. Nodal lines

5. Exotic spectra
   - Mixed spectra
   - Band spectra
   - Landau levels
   - Ten martini problem
I briefly describe five old but still actively explored problems of the Spectral Theory of Partial Differential Equations

1. How eigenvalues are distributed (where eigenvalues often mean squares of the frequencies in the mechanical or electromagnetic problems or energy levels in the quantum mechanics models) and the relation to the behaviour of the billiard trajectories.

2. Equidistribution of eigenfunctions and connection to ergodicity of billiard trajectories (a quantum quantum ergodicity and a classical quantum ergodicity).

3. Can one hear the shape of the drum?

4. Nodal lines and Chladni plates.

5. Strange spectra of quantum systems.
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Reminder

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For operators in the Banach space there are:

1. **Point spectrum**—see above.

2. **Continuous spectrum**: \( \lambda \) is not an eigenvalue, the range of \( (H - \lambda) \) is dense, but \( (H - \lambda)^{-1} \) is an unbounded operator.

3. **Residue spectrum**—the rest: i.e. \( \lambda \) is not an eigenvalue, and the range of \( (H - \lambda) \) is not dense.

However, we are interested in the different classification for self-adjoint operators in the Hilbert space. Point spectrum is defined in the same way, residue spectrum is empty, but continuous spectrum is defined differently and there are two types of it.
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For self-adjoint operators in the Hilbert space there is a spectral decomposition: a family of orthogonal spectral projectors $E(\lambda)$, s.t. $E(\lambda)E(\lambda') = E(\lambda')$ for $\lambda' < \lambda$, $E(-\infty) = 0$, $E(\infty) = I$, $E(\lambda)$ is semi-continuous from the left, i.e. $E(\lambda^-) = E(\lambda)$, and

$$\int \lambda d\lambda E(\lambda) = H.$$
Furthermore, Hilbert space $H$ is decomposed into a direct sum $H = H_{pp} \oplus H_{ac} \oplus H_{sc}$, s.t. projectors $E(\lambda)$ maps each of them into itself, then operator $H$ is decomposed into a direct sum $H = H_p \oplus H_{ac} \oplus H_{sc}$, and restriction of $E(\lambda)$ to $H_{pp}, H_{ac}$ and $H_{sc}$ is an atomic measure, absolutely continuous measure and singular continuous measure respectively. Then the spectra of operators $H_{ac}$ and $H_{sc}$ is called absolutely continuous spectrum and singular continuous spectrum of operator $H$ respectively. Continuous spectrum is an union of the absolutely continuous and singular continuous spectra. And the spectrum of $H_{pp}$ is a closure of the set of eigenvalues (which is not necessarily closed). We call it (not universally accepted) purely point spectrum.
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The definition of the multiplicity of the eigenvalue is wide-known (the dimension of the corresponding eigenspace) and one can define the multiplicity of the absolutely continuous spectrum. **Discrete spectrum** of operator $H$ is a set of its eigenvalues of the finite multiplicity, which are isolated from the rest of the spectrum. The remaining spectrum (eigenvalues of the infinite multiplicity, accumulation points of the point spectrum, points of the continuous spectrum) make an **essential spectrum** of operator $H$. 
In 1911 a 26-years old mathematician, a former student of David Hilbert, Hermann Weyl published a very important paper "Über die asymptotische Verteilung der Eigenwerte" (About the asymptotic distribution of eigenvalues), which followed by four more papers in 1912, 1913 and 1915.
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According to the book of Lord Rayleigh The Theory of Sound (1887) eigenvalues were the squares of the frequencies; in the Quantum Mechanics (developed a bit later) eigenvalues were primarily the energy levels.
For Dirichlet Laplacian in a bounded $d$-dimensional domain $\Omega$ H.Weyl (1911) proved that

$$N(\lambda) = c_0 \mes(\Omega) \lambda^{\frac{d}{2}} + o(\lambda^{\frac{d}{2}})$$

as $\lambda \to +\infty$ and

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$$N(\lambda) = c_0 \text{mes}(\Omega) \lambda^\frac{d}{2} \mp c_1 \text{mes}_{d-1}(\partial \Omega) \lambda^\frac{d-1}{2} + o(\lambda^\frac{d-1}{2}).$$

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In this framework eigenvalue is the non-trivial solution to the equation

$$-\Delta \psi_n := -\left(\partial_{x_1}^2 + \partial_{x_2}^2 + \ldots + \partial_{x_d}^2\right)\psi_n = \lambda_n \psi_n,$$ \hspace{1cm} (3)$$

satisfying the boundary conditions (Dirichlet or Neumann).
To explain (1) consider a rectangular box of the size $a_1 \times a_2$ (for a simplicity we take $d = 2$). Then the eigenvalue problem could be solved easily by the separation of the variables
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\Theta = \{(m_1, m_2) \in \mathbb{Z}^+^2, \frac{m_1^2}{a_1^2} + \frac{m_2^2}{a_2^2} < \frac{\lambda}{\pi^2}\},
$$

which should be approximately equal to the volume of the domain $\Theta$ (which is $1/4$ of the ellipse with semiaxis $\pi - 1 \lambda^{1/2} a_1$, $\pi - 1 \lambda^{1/2} a_2$) i.e.

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\omega_d^2 (2\pi)^d - 1 \lambda d^2 \text{mes}(\Omega) = \omega_d^2 (2\pi)^d - 2 \lambda d^2 \text{mes}(\Omega),
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$$\omega_2 (2\pi)^{-1} \lambda^\frac{1}{2} a_1 \times (2\pi)^{-1} \lambda^\frac{1}{2} a_2 = \omega_2 (2\pi)^{-2} \lambda^\frac{d}{2} \text{mes}(\Omega),$$

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Weyl conjecture (2) was the result of a more precise analysis of the same problem: for Dirichlet boundary condition we do not count points with \( m_i = 0 \) (blue) and for Neumann problem we count them, so in (2) will be “−” for Dirichlet, and “+” for Neumann, \( c_1 = \frac{1}{4} (2\pi)^{1-d} \omega_{d-1} \).
The proof of (1) by Weyl was based on this formula for boxes, integral equations and variational arguments he invented. Those arguments are based on the formula

$$N(\lambda) = \max \dim \mathcal{L},$$

(5)

where $\mathcal{L}$ runs over all subspaces of $\mathcal{H}$ on which quadratic form $\|\nabla u\|^2 - \lambda \|u\|^2$ is negative and $\mathcal{H}$ is Sobolev space $H^1(\Omega)$ in the case of Neumann boundary problem and $H^1_0(\Omega) = \{ u \in H^1(\Omega), u|_{\partial\Omega} = 0 \}$ in the case of Dirichlet boundary problem.
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Richard Courant in 1920, pushing this method to its limits proved remainder estimate to \( O(\lambda \frac{d-1}{2} \log \lambda) \) for bounded domains with \( \mathcal{C}^\infty \) boundary.

Actually both H. Weyl and R. Courant considered only \( d = 2, 3 \).
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In these papers much more general problems were considered and much more subtle arguments were applied. However these methods were not sufficient to prove even $O(\lambda^{(d-1)/2})$ remainder estimate, leave alone Weyl’s conjecture.
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\text{Tr}(f(-\Delta, t)) = \int u(x, x, t) \, dx.
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On the other hand, it is related to \( N(\lambda) \) by

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One of the popular choices is $f(-\Delta, t) = e^{t\Delta}$; then $u$ is a fundamental solution to heat equation, i.e.

$$u_t = \Delta u,$$

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$$u|_{t=0} = \delta(x - y).$$

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So, we have heat equation method. Such function $u(x, y, t)$ is easy to construct but it is very difficult to prove a good remainder estimate from Laplace transform $\sigma(t) := \int e^{-\lambda t} d\lambda N(\lambda)$.

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In this case the recovery part is easy, and one can get a good remainder estimate, but the PDE part is difficult near the border or for large \( t \).
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In this case the recovery part is easy, and one can get a good remainder estimate, but the PDE part is difficult near the border or for large \( t \).

Using construction for \( |t| \leq T \) with some constant \( > 0 \) Levitan recovered remainder estimate \( O(\lambda^{(d-1)/2}) \) for compact closed manifolds (closed = without the boundary).
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However estimate $o(\lambda^{(d-1)/2})$ may fail without some additional condition. Indeed, consider Laplace operator on the sphere $S^d$, for $d = 2$. Then eigenvalues are $\lambda_m = m(m + 1)/2$ of multiplicity $2m + 1 \asymp \lambda_m^{1/2}$ and we cannot get a remainder estimate better than $O(\lambda^{(d-1)/2})$, and the same is true for any dimension.
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Geodesics could be marked by their initial points and directions, which are elements of the phase space, and there is a standard volume measure on the phase space. On the sphere all geodesics (large circles) are periodic, but there are other manifolds (f.e. Zoll-Tanner manifolds) with the same property.
The result of J. J. Duistermaat and Victor Guillemin is remarkable, because it connects the spectral properties of operator (quantum property) and the non-periodicity properties of the Hamiltonian (in this case geodesic flow) $\Psi_t$ (classical property).
The result of J. J. Duistermaat and Victor Guillemin is remarkable, because it connects the spectral properties of operator (quantum property) and the non-periodicity properties of the Hamiltonian (in this case geodesic flow) $\Psi_t$ (classical property). But what about sharp remainder estimate for manifolds (or domains) with the boundary? It is coming!
In 1978 Robert Seeley proved remainder estimate $O(\lambda^{(d-1)/2})$ for compact domains (generalization to manifolds with the boundary was easy).
In 1978 Robert Seeley proved remainder estimate $O(\lambda^{(d-1)/2})$ for compact domains (generalization to manifolds with the boundary was easy). He did not construct explicitly $u(x, y, t)$ near the border for the wave equation in the general case (and nobody did so far!) but he invented some kind of the short dynamics arguments which allowed him to get around this obstacle rather than overcome it.
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If you are interested in this topic (and in the further developments—and there were a lot!), you can find more in my talk 100 years of Weyl’s law and in my article, also called 100 years of Weyl’s law.
In 1974 Alexander Shnirelman began to study the equidistribution of eigenfunctions (of the Laplacian on the closed manifold). Recall that eigenfunctions are functions $\psi_n \neq 0$ such that

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Equidistribution of eigenfunctions

In 1974 Alexander Shnirelman began to study the equidistribution of eigenfunctions (of the Laplacian on the closed manifold). Recall that eigenfunctions are functions $\psi_n \neq 0$ such that

$$-\Delta \psi_n = \lambda_n \psi_n. \quad (3)$$

**Definition 2**

Eigenfunctions are **equidistributed** if

$$\int_{\omega} |\psi_n|^2 \, dx \to \frac{\text{mes}(\omega)}{\text{mes}(\Omega)} \quad \text{as} \quad n \to \infty \quad (14)$$

for all $n$ except those belonging to subsequence $n_k$ of the density 0.
Subsequence $n_k$ has density 0 if it becomes rarified on the long intervals: 
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In fact, Shnirelman was interested in the stronger property: equidistribution in the phase space (space of coordinates and momenta) rather than in the configuration space (space of coordinates only).

It turned out that the equidistribution (which is now called \textit{quantum ergodicity}) is due to the ergodicity of the classical dynamical system, the study of quantizations of classically chaotic systems is sometimes called quantum chaos.
Recall that we have a geodesic flow $\Psi_t$. Due to Liouville’s theorem it preserves the volume of the phase space: $\text{mes}(\Psi_t(\omega)) = \text{mes}(\omega)$ for all $t$. 

**Definition 3**
The flow $\Psi_t$ is called ergodic if for almost all trajectories and all subsets $\omega$ of the phase space $\Omega$ (with $\text{mes}(\omega) > 0$) the time trajectory spends inside $\omega$ (if we consider time interval $[0, T]$), divided by $T$ tends to $\text{mes}(\omega)/\text{mes}(\Omega)$ as $T \to +\infty$.

Almost all = except of measure zero.

One can say that ergodicity means that almost every trajectory forgets the past after a while.
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Ergodicity has been a popular topic of the research and now the equidistribution also is. Since the above result has been generalized to manifolds with the boundary:

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**Ergodicity and equidistribution**

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and there is a little doubt that the converse result is also true, let us consider ergodicity of the Euclidean billiards, and its relation to non-periodicity.
Obviously ergodicity $\implies$ nonperiodicity but converse is not true.
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(a) Circular billiard

(b) Rectangular billiard
Obviously ergodicity $\Rightarrow$ nonperiodicity but converse is not true. Indeed, consider circular billiard or rectangular billiard:

(a) Circular billiard

(b) Rectangular billiard

Circular billiard trajectory is periodic if and only if $\alpha/2\pi = m/n$ is rational (irreducible); then billiard trajectory closes after $n$ reflections and $m$ turns around center; otherwise it fills densely (but not uniformly) the ring between two circles.
Rectangular billiard trajectory (with sides $a, b$) is periodic if and only if $\tan(\alpha) : a/b = m/n$ is rational (irreducible); then billiard trajectory closes after $2m$ reflections from the horizontal sides and $2m$ reflections from the vertical sides; otherwise it fills uniformly densely the whole rectangle.
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In both case trajectory “remembers” $\alpha$ and there is no ergodicity.
Figure: Bunimovich Stadium: billiard flow is ergodic: single trajectory is drawn
Still, some billiard trajectories (bouncing balls) are not dense and there is a sequence of eigenfunctions which are not equidistributed. But the set of such trajectories has measure 0 and the sequence of such eigenfunctions has a density 0.
Can one hear the shape of the drum?

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First, mathematicians tried to find spectral invariants—characteristics which must coincide for isospectral domains (domains with the same spectra).

For this they used heat equation method: they considered

$$
\sigma(t) := \text{Tr}(e^{t\Delta}) = \sum_n e^{-\lambda_n t} \quad t > 0.
$$

(15)
It was known that

$$\sigma(t) = c_0 t^{-d/2} + c_1 t^{(1-d)/2} + \ldots \quad \text{as} \quad t \to +0 \quad (16)$$

Here (for $d = 2$) $c_0$ is proportional to the area, $c_1$ to the perimeter, $c_2$ Euler’s characteristic (connected to the number of holes), etc – so those are spectral invariants, and one with the perfect hearing can hear them.
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Here (for \( d = 2 \)) \( c_0 \) is proportional to the area, \( c_1 \) to the perimeter, \( c_2 \) Euler's characteristic (connected to the number of holes), etc – so those are spectral invariants, and one with the perfect hearing can hear them. But the final answer (given in 1992 by Gordon, Webb, and Wolpert) was negative:
One cannot hear the shape of the drum

There exist isospectral but not isometric domains.

Figure: Example of two isospectral domains
One cannot hear the shape of the drum
There exist isospectral but not isometric domains.

Figure: Example of two isospectral domains

Still plenty of questions remain: can we hear the shape of the convex drum? ...
Nodal lines

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In yearly 19-th century, a musician and physicist Ernst Chlani made an experiment shown thus confirming an earlier observation of Robert Hooke. It is now called Chladni plates.

This experiment has been repeated many times (usually by physics professors trying to impress prospective students), with the violin bow replaced by an electric sound speaker.)
When Chladni showed his experiment in Paris, Napoleon set a prize for the best mathematical explanation.
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Marie-Sophie Germain was the only person who submitted the solution (with the correct approach) but it was rejected: the judging commission felt that “the true equations of the movement were not established,” even though “the experiments presented ingenious results.”

Equation she presented led to the eigenvalue problem for the 4-th order equation

\[
(\frac{\partial^4}{\partial x^4} + \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial y^2} + \frac{\partial^4}{\partial y^4}) \phi_n = \lambda \phi_n \quad (17)
\]

with two boundary conditions and the lines on the Chladni plates are nodal lines on which eigenfunction \( \phi(x, y) \) vanishes.
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with two boundary conditions and the lines on the Chladni plates are nodal lines on which eigenfunction \( \psi(x, y) \) vanishes.
Later mathematical interest switched to a simpler problem for a vibrating membrane:

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(\partial^2_x + \partial^2_y) \psi_n = -\lambda \psi_n \quad \text{in} \; \mathcal{D},
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The first question was how many nodal domains are there where a nodal domain is a connected component of \(\{(x, y): \psi_n(x, y) \neq 0\}\). Let the number of nodal domains be \(N(\psi_n)\).
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The first question was how many nodal domains are there where a nodal domain is a connected component of \(\{(x, y): \psi_n(x, y) \neq 0\}\). Let the number of nodal domains be \(N(\psi_n)\).

In one dimensional case \(\psi_n'' = -\lambda\psi_n, \psi_n(0) = \psi_n(a) = 0\) the answer is simple: \(\psi_n(x) = \sin(\pi nx/a)\) and \(N(\psi_n) = n\).
But two-dimensional case is much more difficult! While we know that \( N(\psi_1) = 1 \) and \( 2 \leq N(\psi_n) \leq n \), it is the only easy result. Indeed, consider rectangular domain \( \mathcal{D} = \{(x, y): 0 \leq x \leq a, 0 \leq y \leq b\} \). Then there are eigenfunctions \( \psi_{(m,n)}(x, y) = \sin(\pi mx/a) \sin(\pi ny/b) \) and the nodal lines and domains are shown here:
But two-dimensional case is much more difficult! While we know that \( N(\psi_1) = 1 \) and \( 2 \leq N(\psi_n) \leq n \), it is the only easy result. Indeed, consider rectangular domain \( \mathcal{D} = \{(x, y): 0 \leq x \leq a, 0 \leq y \leq b\} \). Then there are eigenfunctions \( \psi_{(m,n)}(x, y) = \sin(\pi mx/a) \sin(\pi ny/b) \) and the nodal lines and domains are shown here:

(we assume that the corresponding eigenvalue \( \lambda_{(m,n)} = \pi^2(m^2/a^2 + n^2/b^2) \) is simple).
But if we slightly perturb the rectangular domain (so it will not be a rectangular anymore) the number of the nodal domains decreases because for generic domains there are no intersections: each of them breaks in two possible ways, opening a passage.
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There are other problems: describe the size of the nodal set \( \{(x, y): \psi(x, y) = 0\} \) (a spectacular progress here was recently made), how nodal lines meet the boundary and so on.
Exotic spectra: mixed spectra

We considered the cases when the spectrum consists of the eigenvalues of the finite multiplicity (discrete spectrum). But in the self-adjoint operators of mathematical physics can have other kinds of the spectra.
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Schrödinger operator \( H = -\frac{1}{2m} \hbar^2 \Delta + V \)

for the free particle (\( V = 0 \)) has an absolutely continuous spectrum \([0, \infty)\).
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Schrödinger operator \( H = -\frac{1}{2m}\hbar^2 \Delta + V \)

for the free particle \((V = 0)\) has an absolutely continuous spectrum \([0, \infty)\). In the general case, it has an absolutely continuous spectrum \([0, \infty)\) and eigenvalues in \((-\infty, 0)\); if \(V\) decays fast at infinity, there are finite number of eigenvalues, but for Coulomb potential these eigenvalues accumulate to \(-0\).
Dirac operator $H = \sum_\nu \gamma_\nu (-i\hbar \partial_\nu) + \gamma_0 mc^2 + V$

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With the potential, decaying at infinity, it has a finite or infinite number of eigenvalues in the spectral gap; they may accumulate only to the ends of the gap.
Band spectra

Schrödinger operator with periodic potential $V$

has a band spectrum: bands of absolutely continuous spectrum are separated by spectral gaps.

In dimension $d = 1$ for generic $V$ there is an infinite number of gaps.
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In dimension $d = 1$ for generic $V$ there is an infinite number of gaps. For $d \geq 2$ there is only a finite number of gaps—Bethe-Sommerfeld conjecture, proved in full generality only about 10 years ago by Leonid Parnovski and Alexander Sobolev.
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Adding another potential \( W \) decaying at infinity, one can place a finite or infinite number of eigenvalues inside each spectral gap.
Integrated Density of States

Instead of eigenvalue counting function $N(\lambda)$ in this case use Integrated Density of States

$$N(\lambda) := \lim_{\ell \to \infty} \frac{N(\lambda, \ell X)}{\text{mes}(\ell X)}$$

where $0 \in X$ is an open bounded domain in $\mathbb{R}^d$ and $N(\lambda, \ell X)$ is an eigenvalue counting function for the same operator in $\ell X$ (stretched $X$) with the Dirichlet boundary condition.
It turns out that there is a complete spectral asymptotics (even if $V$ is only almost periodic)

$$N(\lambda) \sim \sum_{n=0}^{\infty} \kappa_n \lambda^{d/2-n} \quad \text{as} \quad \lambda \to +\infty.$$ 

Again, this is a rather new result, due to Leonid Parnovski and Roman Shterenberg.
Landau levels

2D Schrödinger operator with constant magnetic field

\[ H = \frac{1}{2m}(-i\hbar\partial_1 - Bx_2/2)^2 + (-i\hbar\partial_2 + Bx_1/2)^2 + V \]

for a free particle \((V = 0)\) has a pure point spectrum of infinite multiplicity, consisting of Landau levels 

\[ E_n := \frac{1}{2m}(2n + 1)B\hbar, \]

\(n = 0, 1, 2, \ldots:\)
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Let \(V\) decay at infinity. Then the spectrum consists of eigenvalues \(e_{n,k}\)

\[ e_{n,k} \to E_n \text{ as } k \to \infty: \]

depending on \(V\) these eigenvalues accumulate to Landau levels either from the left, or from the right, or from the left and right.
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**Essential spectrum** consists of Landau levels.
Classification of spectra

The classification of the spectrum is based on the spectral measure. Each measure $\mu$ could be decomposed into the sum of three measures:

- **atomic measure**, supported in the finite or enumerable number of points (point spectrum) with $\mu(\lambda) := \mu((−\infty, \lambda)) = \sum_{k, \lambda_k < \lambda} m_k$,
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- **atomic measure**, supported in the finite or enumerable number of points (point spectrum) with $\mu(\lambda) := \mu((\infty, \lambda)) = \sum_{k, \lambda_k < \lambda} m_k$,

- **absolute continuous measure** with $\mu(\lambda) = \int_{-\infty}^{\lambda} \rho(t) \, dt$, where $\rho$ is the density function.

- **singular continuous measure** with $\mu(\lambda)$ being continuous, but with $\mu'(\lambda) = 0$ almost everywhere.
Classification of spectra

The classification of the spectrum is based on the spectral measure. Each measure $\mu$ could be decomposed into the sum of three measures:

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And until recently in all “real life” examples the singular continuous spectrum was empty.
Ten martini problem

But there was a candidate, almost Mathieu operator which is the discrete Schrödinger operator (on $\mathbb{Z}$)

$$ (H_{\lambda,\alpha,\theta}\psi)_n = \psi_{n+1} + \psi_{n-1} + 2\lambda \cos(2\pi(\theta + n\alpha))\psi_n, \quad (20) $$

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which appears in mathematical study of the quantum Hall effect. For rational $\alpha$ this operator would have a band spectrum, but for irrational $\alpha$ it was conjectured by Mark Kac that it has a spectrum which is Cantor set. And he offered ten bottles of Martini to one who would prove it!
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$$(H_{\lambda, \alpha, \theta, \psi})_n = \psi_{n+1} + \psi_{n-1} + 2\lambda \cos(2\pi(\theta + n\alpha))\psi_n,$$  \hspace{1cm} (20)

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In 2009 Arthur Avila and Svetlana Zhitomirskaya solved this problem, proving that for all $\lambda \neq 0$ and irrational $\alpha$ the spectrum is a Cantor set.
Cantor set is a closed set which is nowhere dense. However such sets could have positive measures. It is known that the spectrum does not depend on $\theta$.

It is now known, that

- For $0 < \lambda < 1$, $H_{\lambda,\alpha,\theta}$ has surely purely absolutely continuous spectrum.
- For $\lambda = 1$, $H_{\lambda,\alpha,\theta}$ has almost surely purely singular continuous spectrum. (It is not known whether eigenvalues can exist for exceptional parameters.)
- For $\lambda > 1$, $H_{\lambda,\alpha,\theta}$ has almost surely pure point spectrum. It is known that almost surely cannot be replaced by surely.
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That’s all!
Thank you!!