Eigenvalue Asymptotics for Fractional Laplacians

Partial Differential Equations and Analysis Seminar
Mathematical Sciences Institute, Australian National University

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Introduction

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At Spring 2016 my son Oleg mentioned in the Google chat that he would like to learn an analytic technique developed by “one cool guy” who actually refers to one of my papers.

The “cool guy” turned out to be Professor Rodrigo Bañuelos from Purdue University (and indeed, he is really cool) who is doing some probabilistic research and is interested in the spectral asymptotics for generators of some unitary group describing random walks in domain with boundary and and believing that it would be nice to get “Ivrii-type” results for such operators.

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Finally, the colleague whom I knew (since his field was much closer to mine) Rupert Frank from Caltech and from whose paper with Leander Geisinger I was able to learn what exactly is this operator–defined in the way I could work with.

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Here we go...

Operator

Let $X \subset \mathbb{R}^d$, $d \geq 1$ (but $d = 1$ has beed explored and there is no two term asymptotics here anyway) be a compact domain with the smooth boundary $\partial X \in \mathscr{C}^\infty$ and let $(g^{jk}(x))$ be a smooth non-degenerate Riemannian metrics defined in $\mathbb{R}^d$ (actually how exactly we extend it from $\bar{X}$ does not matter).

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Consider pseudo-differential operator $\Delta^{m/2}$ which is Weyl quantization of $g(x,\xi)^{m/2}$, with $m > 0$ (so, it is self-adjoint) and consider the corresponding operator in X with the Dirichlet boundary condition:

$$\Lambda_{m,X} : \mathscr{H}_0^{m/2}(X) \to \mathscr{H}^{-m/2} \qquad \Lambda_{m,X} u := \bigl(\Delta^m u\bigr)\bigr|_X \tag{1}$$

where $\mathscr{H}_0^{m/2}(X) := \{u \in \mathscr{H}^{m/2}(\mathbb{R}^d) : \operatorname{supp}(u) \subset \bar{X}\}$, $\bar{X}$ is a closure of X and $v|_X$ is a restriction of v to X.

Obviously, $\Lambda_{m,X}$ is a bounded operator from $\mathscr{H}_0^{m/2}(X)$ to $\mathscr{H}^{-m/2} := (\mathscr{H}_0^{m/2})^*$ but it also is a self-adjoint unbounded operator in $\mathscr{L}^2(X)$ with an appropriate domain

$$\mathfrak{D}(\Lambda_{m,X}) := \{u \in \mathscr{H}_0^{m/2}(X) : \Lambda_{m,X} \in \mathscr{L}^2(X)\} \tag{2}$$

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and it differs from the operator $\Delta_X^{m/2}$ which is a power of Δ_X (a positive Laplacian with the Dirichlet boundary condition) and while difference is also of order m it is "concentrated" near ∂X (more about this later) and contributes only to the second term in the asymptotics.

Remark

As Δ^m does not possess transmission property (unless $m \in 2\mathbb{Z}$ and this case is completely covered by an existing theory) neither $\Lambda_{m,X}$ nor Δ_X^m belong to L. Boutet De Monvel's algebra (and $u \in \mathscr{C}^\infty(\bar{X}) \cap \mathfrak{D}(\Lambda_{m,X} u)$ does not imply $\Lambda_{m,X} u \in \mathscr{C}^\infty(\bar{X})$ etc but it does not really matter for our analysis).

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Conjecture

$\Lambda_{m,X} - \Delta_X^{m/2}$ is an integral operator with the Schwartz kernel $K(x,y)$ which in the local coordinates such that X coincides with $\{x:\ x_1>0\}$ and $\{y:\ y_1>0\}$ satisfies

$$|D_x^\alpha D_y^\beta K(x,y)| \le C_{\alpha\beta} x_1^{-\frac{m}{2}-\alpha_1} y_1^{-\frac{m}{2}-\beta_1} (x_1+y_1+|x'-y'|)^{-d-|\alpha'|-|\beta'|}\,. \tag{3}$$

Results

As X is bounded the spectrum of $\Lambda_{m,X}$ is discrete tending to $+\infty$ and let $\mathsf{N}(\lambda) = \mathsf{N}_{m,X}(\lambda)$ be an eigenvalue counting function which is the number of eigenvalues of $\Lambda_{m,X}$ (counting multiplicities) lesser than λ.

Theorem 1

1. *The following asymptotics holds:*

$$\mathsf{N}(\lambda) = \kappa_0 \lambda^{d/m} + O(\lambda^{(d-1)/m}) \qquad \textit{as} \;\; \lambda \to +\infty. \tag{4}$$

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2. *If $d \geq 2$ and the measure of periodic geodesic billiards is zero then two-term asymptotics holds:*

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Here

$$\kappa_0=(2\pi)^{-d}\omega_d\,\mathsf{vol}(X) \tag{6}$$

is the standard coefficient, ω_k is a volume of the unit ball in $\mathbb{R}^k$, $\mathsf{vol}(X)$ is a Riemannian volume of X and coefficient κ_1 will be given by (8)–(9).

1D operator and κ_1

Let us consider 1-dimensional toy-model: $X = \mathbb{R}^+$ is a half-line with the Euclidean metrics. Consider operator

$$\mathbf{a}_m = ((D_x^2 + 1)^{m/2})_{\mathrm{D}} \tag{7}$$

and denote by $\mathbf{e}_m(x_1, y_1, \lambda)$ the Schwartz kernel of its spectral projector.

Proposition 2

$\mathbf{a}_m$ *has absolutely continuos spectrum.*

We can express κ_1 in terms of $\mathbf{e}_m(x,y,\lambda)$:

$$\kappa_{1,m} = (2\pi)^{1-d}\omega_{d-1}\varkappa_m \operatorname{vol}_{d-1}(\partial X) \tag{8}$$

where

$$\varkappa_m = \\ = \frac{d-1}{m}\iint_1^\infty \lambda^{-(d-1)/m-1}\Bigl(\mathbf{e}_m(x_1,x_1,\lambda) - \pi^{-1}(\lambda-1)^{1/m}\Bigr)\,dx_1 d\lambda \tag{9}$$

where $\pi^{-1}(\lambda-1)^{1/m}$ is a Weyl approximation to $\mathbf{e}_{m,1}(x_1,x_1,\lambda)$.

Proof sketched

Let us apply "Schrödinger equation" approach. We need to calculate the number $\mathsf{N}^-_h(\tau)$ of the eigenvalues not exceeding τ (as $\tau=1$) of the "Schrödinger" operator

$$H = h^m \Lambda_{m,X} - 1, \qquad h = \lambda^{-1/m}. \tag{10}$$

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$$u(x, y, t) = \int_{-\infty}^{\infty} e^{-ih^{-1}t\tau}\, d_\tau e_h(x, y, \tau) \tag{11}$$

and also

$$(-ih\partial_t + H)u = 0, \tag{12}$$

$$u|_{t=0} = \delta(x - y) \tag{13}$$

where by default all operators are acting with respect to x.

Simple rescaling method

One can prove easily that

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$$e(x,x,\tau) = \varkappa(x)\tau^{d/m}h^{-d} + O(\gamma^{-1}h^{1-d}) \qquad \text{as } \ \tau \asymp 1. \tag{15}$$

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with the standard coefficient $\varkappa(x)$. Summation over $\{x:\,\gamma(x)\ge h\}$ of (15) and over $\{x:\,\gamma(x)\le h\}$ of (14) brings

$$\mathsf{N}^-(1) = \int e(x,x,\tau)\,dx = \kappa_0 h^{-d} + O(h^{1-d}\log h) \tag{16}$$

which is not good! Remember, $h = \lambda^{-1/m}$.

Seeley's method

The trouble is a factor γ^{-1} in the remainder estimate (15). However we can apply R. Seeley's method ([RS], 1978) simplified as we are interested only in asymptotics of $e(x,x,\tau)$ integrated with respect to x. Instead of (15) we can use

$$\int \Bigl(e(x,x,\tau)-\varkappa(x)\tau^{d/m}h^{-d}\Bigr)\psi(x)\,dx = O(\gamma^{\delta+d-1}h^{1-d}+\gamma^{d-2}h^{2-d}) \tag{17}$$

as $\tau\asymp 1$ and $\psi\in\mathscr{C}_0^\infty(B(\mathrm{y},\gamma))$ is γ-admissible where $B(\mathrm{y},\gamma)$ is a ball of radius γ with a center at y, $\gamma=\gamma(\mathrm{y})$ and $\tau\asymp 1$.

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$$\int \Bigl(e(x,x,\tau) - \varkappa(x)\tau^{d/m}h^{-d}\Bigr)\psi(x)\,dx = O(\gamma^{\delta+d-1}h^{1-d} + \gamma^{d-2}h^{2-d}) \tag{17}$$

as $\tau \asymp 1$ and $\psi \in \mathscr{C}_0^\infty(B(\mathsf{y},\gamma))$ is γ-admissible where $B(\mathsf{y},\gamma)$ is a ball of radius γ with a center at y, $\gamma=\gamma(\mathsf{y})$ and $\tau \asymp 1$.
To prove (17) we need to remember that γ^{-1} appears in the remainder estimate only because $T \asymp \gamma$ is a time for which we can trace propagation of singularities for (12) and it should be T^{-1}. And because there is mollify integration by x we can choose direction as we wish and we chose away from the boundary and can take as Seeley did $T \asymp \gamma^{-1/2}$.

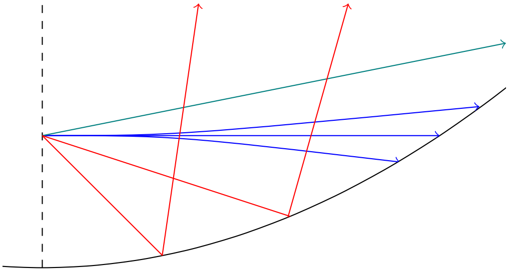

Figure: Green rays are outgoing, red are incoming and blue are almost parallel to the boundary.

On this picture we do not need to consider red rays!

Summation over $\{x : \gamma(x) \geq h\}$ of (16) and over $\{x : \gamma(x) \leq h\}$ of (14) brings remainder estimate $O(h^{1-d})$ i.e. $O(\lambda^{(d-1)/2m})$ i.e. the first statement of Theorem 1.

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But it is not our purpose as we want "Ivrii-type results" and the remainder estimate $O(\lambda^{(d-1)/m})$ one could probably get by variational methods from the same results with $m\in 2\mathbb{Z}$.

For “Ivrii-type results” we need “Ivrii-type methods”. 35+ y.a. (before original Weyl conjecture, i.e. for $m = 1$) was proven the stumbling block were rays (almost) tangent to the boundary and R. Seley and V. Ivrii found different ways to overcome it ($m \in \mathbb{Z}^+$ was an easy generalization). D. Vassiliev had shown a bit later that Seeley’s method allows to derive the second term as well.

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Now those rays could be dealt in the same way but a new stumbling block are rays (almost) normal to the boundary–because operator now does not have transmission property and even if $v\in\mathscr{C}^\infty(\bar{X})\cap\mathscr{C}^m_0(\bar{X})$ we have $\Lambda_{m,X}v$ is not smooth exactly on ∂X in the normal direction.

Propagation near the boundary

Notations

Consider small vicinity of the boundary ∂X; we introduce coordinates $x=(x_1;x')=(x_1;x_2,\ldots,x_d)$ in which $x_1=\mathsf{dist}(x,\partial X)$ and $g^{1j}=\delta_{1j}$ for all j. Let $D_j=-i\partial_j$.

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Then using positive commutator method (with $L=L(x',hD')$) we prove that if $Q=Q(x',hD')$ has a symbol supported in $\{(x',\xi'):\,|\xi'|\ge\epsilon\}$ then singularities of $\varphi(hD_t)\chi_T(t)Q(x',hD')\psi_0(y)u_h$ are supported in $\{|x'-y'|\asymp T\}$ as $T\in[h^{1-\delta},\epsilon_0]$.

Notations

Here and below $\chi\in\mathscr{C}_0^\infty([-1,-\frac{1}{2}]\cup[\frac{1}{2},1])$, $\chi_T(t)=\chi(t/T)$ and $\varphi\in\mathscr{C}^\infty(\mathbb{R})$, $\operatorname{supp}(\varphi)\subset\{\tau\le c\}$, $\psi_0\in\mathscr{C}^\infty(X)$, $\operatorname{supp}(\psi_0)\subset\{x_1\le\epsilon_0\}$, ϵ_j are small constants and $\delta>0$ is an arbitrarily small exponent.

Then

$$F_{t\to h^{-1}\tau}\Gamma_x\bigl(\chi_T(t)Q(x',hD')\psi_0(y)u_h\bigr) = O(h^s) \qquad \text{as } \ \tau \le c \tag{18}$$

where here and below Γ is an operator of the restriction to the diagonal $\{x=y\}$.

Then

$$F_{t\to h^{-1}\tau}\Gamma_x\bigl(\chi_T(t)Q(x',hD')\psi_0(y)u_h\bigr)=O(h^s)\qquad \text{as}\ \ \tau\le c \tag{18}$$

where here and below Γ is an operator of the restriction to the diagonal $\{x=y\}$.This was a part of my approach of 1979 (reformulated several years later).

Propagation near the boundary

Next, using positive commutator method (with $L = x_1 hD_1 - ih/2)$) we prove that for 1-dimensional operator at energy level 1 singularities cannot "stall" near x_1, that they are really reflecting from the boundary.

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Then it is also true for d-dimensional case, Euclidean metrics, $X = \{x : x_1 \geq 0\}$ after cut-off by $Q = Q(x; hD')$ with the symbol supported in $\{|\xi'| \leq (1-\epsilon)\}|\tau|^{1/m}$, $\tau \asymp 1$ is an energy level.

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Then using successive approximation method and propagation of singularities as $x_1 \ge h^{1-\delta}$ we prove that this is also true in the general case as $|\xi'|$ is calculated according to metrics (g^{jk}).

Then in this framework

$$F_{t\to h^{-1}\tau}\Gamma\bigl(\chi_T(t)Q(x',hD')\psi_0(y)u_h\bigr)=O(h^s)\qquad \text{as}\ \ \tau\asymp 1 \tag{19}$$

where $\Gamma v=\int v(x,x,t)\,dx$.

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How does it differ from (18)?

1. First, now Q has a symbol supported in $\{|\xi'|\le (1-\epsilon)|\tau|^{1/m}\}$ rather than in $\{|\xi'|\ge \epsilon|\tau|^{1/m}\}$,

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2. Second, now we need to apply Γ rather than Γ_x.

This analysis is new. As $m \in \mathbb{Z}^+$ it was well known reflection of the transversal rays from the border. Now we cannot calculate reflected wave; we just know that it bounces from ∂X.

Estimates in the boundary zone

Combining (18) and (19) we conclude that (19) holds for $Q = I$. Then using Tauberian method we conclude that as $\operatorname{supp}(\psi_0) \subset \{x_1 \le 2\varepsilon\}$, $\psi_0 \le 1$

$$|\int \bigl(e(x,x,\tau) - e^{\mathsf{T}}(x,x,\tau)\bigr)\psi_0(x)\,dx| \le C_0\varepsilon h^{1-d} + o_\varepsilon(h^{1-d}) \tag{20}$$

where

$$e^{\mathsf{T}}(x,x,\tau) = h^{-1}\int_{-\infty}^{\tau} \bigl(F_{t\to h^{-1}\tau'}\bar\chi_T(t)\Gamma_x u_h\bigr)\,d\tau' \tag{21}$$

is a Tauberian approximation; here and below $\bar\xi \in \mathscr{C}_0^\infty([-1,1])$, $\bar\chi = 1$ on $[-\frac{1}{2},\frac{1}{2}]$ and we can take $T = h^{1-\delta}$; C_0 does not depend on $\varepsilon > 0$ which is an arbitrarily small constant.

Next using successive approximation method we can prove

$$|\int \bigl(e^{\mathsf{T}}(x,x,\tau)-\tau^d h^{-d}\varkappa_0(x)\bigr)\psi_0(x)\,dx - \underbrace{\int_{\partial X}\tau^{d-1}h^{1-d}\varkappa_1(x')\psi_0(x)\,dx'}\,| \le C_0\varepsilon h^{1-d}+o_\varepsilon(h^{1-d}) \tag{22}$$

where

$$\varkappa_1(x') = \int_0^\infty \mathbf{e}_{m,d}(x',x_1;x',x_1;1)\,dx_1 \tag{23}$$

and $\mathbf{e}_{m,d}(x,x,\tau)$ is a Schwartz kernel of the spectral projector of d-dimensional operator in the half-space $\{x:\ x_1>0\}$; as $\psi_0=1$ on ∂X and $\tau=1$ the selected term coincides with $\kappa_1 h^{1-d}$ defined by (8)–(9).

Combining (20) and (22)–(23) we arrive to

Proposition 3

Estimate (22) holds for $e(x, x, \tau)$ as well (not just for its Tauberian approximation).

Contribution of the inner zone

Now we need to estimate

$$\int \bigl(e(x,x,\tau) - h^{-d}\varkappa_0(x)\bigr)\psi_1(x)\,dx_1 \tag{24}$$

where $\psi_1 \in \mathscr{C}_0^\infty(X)$ is supported in $\{x:\, x_1 \ge \varepsilon/2\}$.

Proposition 4

If the measure of periodic geodesic billiards is zero then expression (24) is $o(h^{1-d})$.

Proof.

The proof is standard (as in the original Weyl conjecture) as long as we know that singularities propagate along geodesic billiards transversal to the boundary. We do not need to consider now generalized billiards tangent to the boundary at some points.

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Proof of Theorem 1.

Theorem 1 is due to Propositions 3 and 4. □

Further discussion

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$$\operatorname{dist}(x,y) \le C_0|x-y| \qquad \forall x,y\in X \tag{25}$$

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4. Generalizations to domains described in [MonsterBook], Section 12.1–with thick cusps–seems to be a daunting task!

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$$\tau^{-1}\operatorname{Tr}((\tau - \Lambda_{m,X})^+) = \frac{m}{d+m}\kappa_0\tau^{d/m} + \frac{m}{d+m-1}\kappa_1\tau^{(d-1)/m} + O(\tau^{(d-2)/m}) \qquad (26)$$

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- Chapter 7. Standard Local Semiclassical Spectral Asymptotics near the Boundary.
- Section 8.5. Fractional Laplacians.
- Section 11.2. Large eigenvalues for operators with weakly singular potentials.
- Section 12.1. Operators in domains with thick cusps.

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