Talk 1:

Sharp Spectral Asymptotics for Irregular Operators

Victor Ivrii,

Department of Mathematics,

University of Toronto

2003

1 Problem and Results

1.1 Preface

This talk presents rather old results. Part of them I obtained in Spring 1999 when I was Lady Davies Fellow at Technion (Haifa), part I obtained together with Michael Bronstein (Kazan State Technological University, Russia) when he visited Toronto at Fall 2000.

The results are now published in our paper

Sharp Spectral Asymptotics for Operators with Irregular Coefficients. I. Pushing the Limits. Comm. Part. Diff. Equats., v. 28, no 1&2, pp. 99-123, 2003.

The further progress is heavily based on this paper.

Physicists: talk about things which they believe are true	**vs**	**Mathematicians**: talk about things which they believe they proved

These methods (in simpler form) work well to justify Thomas-Fermi theory (with all corrections - Scott and Dirac-Schwinger) starting from Multidimensional Schrödinger Operator if magnetic field is not too strong

Historical Insert

For Laplacian H.Weyl (1911) proved that

$$N(\lambda) = c_0 \lambda^{\frac{d}{2}} + o(\lambda^{\frac{d}{2}}) \qquad (*)$$

as $\lambda \to +\infty$ and conjectured that

$$N(\lambda) = c_0 \lambda^{\frac{d}{2}} + c_1 \lambda^{\frac{d-1}{2}} + o(\lambda^{\frac{d-1}{2}}) \qquad (WC)$$

R.Courant (1924):

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B.Levitan (1952) and V.Avakumovič (1956) proved that if there is no boundary then

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R.Seeley (1978): (***) with the boundary!!!!

V.Ivrii (1979): Weyl conjecture proven! Geometric condition: Periodic geodesic billiards have measure 0.

\begin{modestyoff}

Soon it will be 25-th anniversary.

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All papers from LA required very large smoothness

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S.Fedorova-V.Ivrii, V.Ivrii (1986): Angles, vertices, edges, conical points and other “rare” singularities allowed.

Sharp estimates in the smooth case.	vs	Bad estimates in the general case.

1.2 Problem

I will deal with Laplace

$$L = \sum_{j,k} D_j g^{jk} D_k \tag{1}$$

and Schrödinger

$$A_h = h^2 \sum_{j,k} D_j g^{jk} D_k + V \tag{2}$$

operators (with real-valued coefficients, symmetric positive definite matrix (g^{jk}) but results and approach remain for higher-order and matrix operators as well.

I am interested in asymptotics of eigenvalue counting functions $N(\lambda) = \#\{\mu_j \in \mathsf{Spec}\ (L), \mu_j < \lambda\}$ as $\lambda \to +\infty$ and $N(h) = \#\{\mu_j \in \mathsf{Spec}\ (A_h), \mu_j < 0\}$ as $h \to +0$.

The second, semiclassical asymptotics is more universal: due to Birman-Schwinger principle $N(\lambda) = \#\{\mu_j \in \mathsf{Spec}\ (\frac{1}{\lambda} L - I), \mu < 0\}$ and we need just to pick $h = 1/\sqrt{\lambda}$, $V = -1$.

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In the smooth case the best possible result is

$$N_h = \mathsf{Weyl} + O(h^{1-d}) \tag{3}$$

where

$$\mathsf{Weyl} = (2\pi h)^{-d} \iint_{a(x,\xi)<0} dx\, d\xi \tag{4}$$

with $a(x,\xi) = \sum_{j,k} g^{jk}\xi_j\xi_k + V$

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I am not talking about boundary now because in what I am doing boundary is rather minor annoyance than major obstacle. So, I am considering operator on compact closed manifold (in fact I consider $\int e_h(x,x,0)\psi(x)\,dx$ where $e_h(x,y,\tau)$ is a Schwartz kernel of spectral projector of A_h and $\psi(x)$ is a cut-off function).

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What are smoothness condition to g^{jk}, V sufficient for asymptotics (3)-(4)?

I want remainder estimate $O(h^{1-d})$ or $o(h^{1-d})$ and want to weaken as much as possible the smoothness condition.

1.3 Approach

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Mollify!

If we mollify coefficients by taking a convolution with ε^{-d}æ(x/ε) where æ is a smooth function with $\int$ æ$x^\alpha(x)\,dx = \delta_{\alpha\,0}$ then the mollification error in g^{jk}, V will be $\vartheta(\varepsilon)$ provided $\nu(\varepsilon) = \varepsilon^{-1}\vartheta(\varepsilon)$ is a continuity modulus of first derivatives of g^{jk}, V.

So, we assume that

$g^{jk}, V \in C^1$ and their derivatives satisfy some continuity condition:

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We do not a need stronger assumptions and cannot derive (3)-(4) under weaker one.

Then under reasonable assumption approximation error in Weyl does not exceed $Ch^{-d}\vartheta(\varepsilon)$ and to include it in $O(h^{1-d})$ one needs to assume that

$$\vartheta(\varepsilon) = O(h) \tag{5}$$

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1.4 Fast Charge . . . and Failure

After we mollified coefficients we can rescale $x \mapsto x/\varepsilon$ and in the new coordinates coefficients will be uniformly smooth and then the remainder estimate should be $O(h_{\text{eff}}^{1-d}\varepsilon^{-d}) = O(h^{1-d}\varepsilon^{-1})$ where $h_{\text{eff}} = h/\varepsilon$ is "new" h (after rescaling) and ε^{-d} in a Jacobian.

So, one can try to pick up parameter ε minimizing approximation error + remainder estimate

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1.5 Recovery

Well, our fast charge failed. Let's look: we applied rescaled standard results. Standard results are based on the Tauberian approach: we consider $u_h(x,y,t)$ the Schwartz kernel of $e^{ih^{-1}tA_h}$. Here and below A means mollified operator, therefore A, $u_h(x,y,t)$, $e_h(x,y,\tau)$ depend also on ε. Actually we take two approximate operators $A^{\pm}_\varepsilon$ framing A: $A^-_\varepsilon \le A \le A^+_\varepsilon$ in operator sense.

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Therefore in rescaled smooth theory $T=T_0=\epsilon\cdot\varepsilon$ and to get desired remainder estimate $O(h^{1-d})$ we need to get back $T=T_1=\epsilon$.

We want to prove that

$$\phi(hD_t)\int u_h(x,x,t)\psi(x)\,dx = O(h^s) \qquad\qquad T_0 \le |t| \le T_1. \qquad (8)$$

To do this we need to study propagation of singularities, prove that they propagate along Hamiltonian trajectories and that these trajectories leave diagonal (everything should be done in our settings and in an appropriate sense).

1.6 Propagation of Singularities

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(a) Oscillatory integrals,

(b) Heisenberg transformation of operators,

(c) Energy estimates.

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Two first approaches have the same shortcomings: they cannot handle well general systems and in these approaches variables x and ξ play the same role (because on time interval of the finite length these variables are "mixed"). So, in these approaches

$$\boxed{\text{scale with respect to } x} = \boxed{\text{scale with respect to } \xi} \tag{9}$$

and we need to fulfil at least uncertainty principle

$$\boxed{\text{scale with respect to } x} \times \boxed{\text{scale with respect to } \xi} \geq h \tag{10}$$

and since

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Actually, uncertainty principle as formulated above, is too weak for Microlocal Analysis which require at least logarithmic uncertainty principle

$$\boxed{\text{scale with respect to } x} \times \boxed{\text{scale with respect to } \xi} \geq Ch|\log h|. \tag{12}$$

We use approach (c) in which evil condition (9) is replaced by

$$\boxed{\text{scale with respect to } \xi} = \epsilon \tag{13}$$

and then from logarithmic uncertainty principle we get

$$\varepsilon = C_0 h |\log h| \tag{14}$$

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YES, IT IS!!!

This what my talk is about.

1.7 A Bit More of Heuristics

If V is disjoint from 0 then on energy level 0 also $|\xi|$ is disjoint from 0. Then for time T shift with respect to x is $\asymp T$. We want to have it observable, so

$$T \ \times \boxed{\text{scale with respect to } \xi} \geq Ch|\log h| \tag{15}$$

and therefore due to (13) $T = T_0 = \epsilon\varepsilon$ (provided constant C_0 in definition is large enough). I remind, that T_0 came from rescaling of smooth results.

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It does not work well other way: if $\nabla_x a(x,\xi)$ is disjoint from 0 the shift with respect to ξ is $\asymp T$ but condition

$$T \quad \times \boxed{\text{scale with respect to } x} \ge Ch|\log h| \tag{16}$$

requires $T \ge Ch\varepsilon^{-1}|\log h|$ which is much more restrictive.

1.8 Results

Theorem 1 *Let* (g^{jk}) *be symmetric positive definite matrix. Let first derivatives of* g^{jk}, V *be continuous with continuity modulus* $|\log|x-y||^{-1}$. *Let*

$$V \leq -\epsilon. \tag{17}$$

Then asymptotics (3)-(4) holds.

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Then asymptotics (3)-(4) holds.

$$N_h = \mathsf{Weyl} + O(h^{1-d}) \tag{3}$$

where

$$\mathsf{Weyl} = (2\pi h)^{-d} \iint_{a(x,\xi)<0} dx\,d\xi \tag{4}$$

How about remainder estimate $o(h^{1-d})$?

It requires condition to Hamiltonian flow Φ_t defined by

$$\frac{dx}{dt} = \partial_\xi a(x,\xi), \qquad \frac{d\xi}{dt} = -\partial_x a(x,\xi). \tag{18}$$

Unless g^{jk}, V have bounded second derivatives, $\partial_x a(x,\xi)$ does not satisfy Lipschitz' condition and Hamiltonian flow is multi-valued. Then we call point (x,ξ) periodic if $\exists t\ne 0 : (x,\xi)\in \Phi_t((x,\xi))$.

Theorem 2 *Let conditions of Theorem 1 be fulfilled. Let first derivatives of* g^{jk}, V *be continuous with continuity modulus* $o(|\log|x-y||^{-1})$. *Further, let the set of periodic points of* Φ *have measure* 0. *Then asymptotics*

$$N_h = \mathsf{Weyl} + o(h^{1-d}) \tag{19}$$

holds.

To consider manifolds with boundaries we need to replace trajectories by billiards and to assume additionally that the set of dead-end points has measure 0 as well. So, unless we assume that g^{jk}, V have bounded second derivatives, checking these assumptions is virtually impossible.

1.9 Discussion

What about stronger singularities?

I treated such singularities concentrated on manifolds of codimension 2 17 years ago (*). Now I consider weaker singularities, but

They are everywhere!!!

However, the same rescaling technique as in (*) could be applied here as well and we can consider strong singularities concentrated on manifolds of codimension 2 on the top of general weak irregularity. Therefore counter-examples showing that one cannot weaken conditions of theorem 1, are very difficult - one needs to consider irregularities everywhere.

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Do not ask for counter – examples – I do not have them!

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In dimensions $1, 2$ we can trade them to condition

$$|V| + |\nabla V| \geq \epsilon \tag{20}$$

and while in dimension 2 it is done by rescaling technique only, dimension 1 requires more delicate analysis.

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Details are published my paper (see below).

1.10 Boundary

As I mentioned, boundary does not pose a big problem. Consider point x on the distance $\gamma(x)$ from the boundary. Rescaling ball $B(x, \frac{1}{2}\gamma(x))$ into $B(0,1)$ we see that $h_{\mathsf{eff}} = h/\gamma$ and its contribution to the remainder estimate is $O(h^{1-d}/\gamma(x))$.

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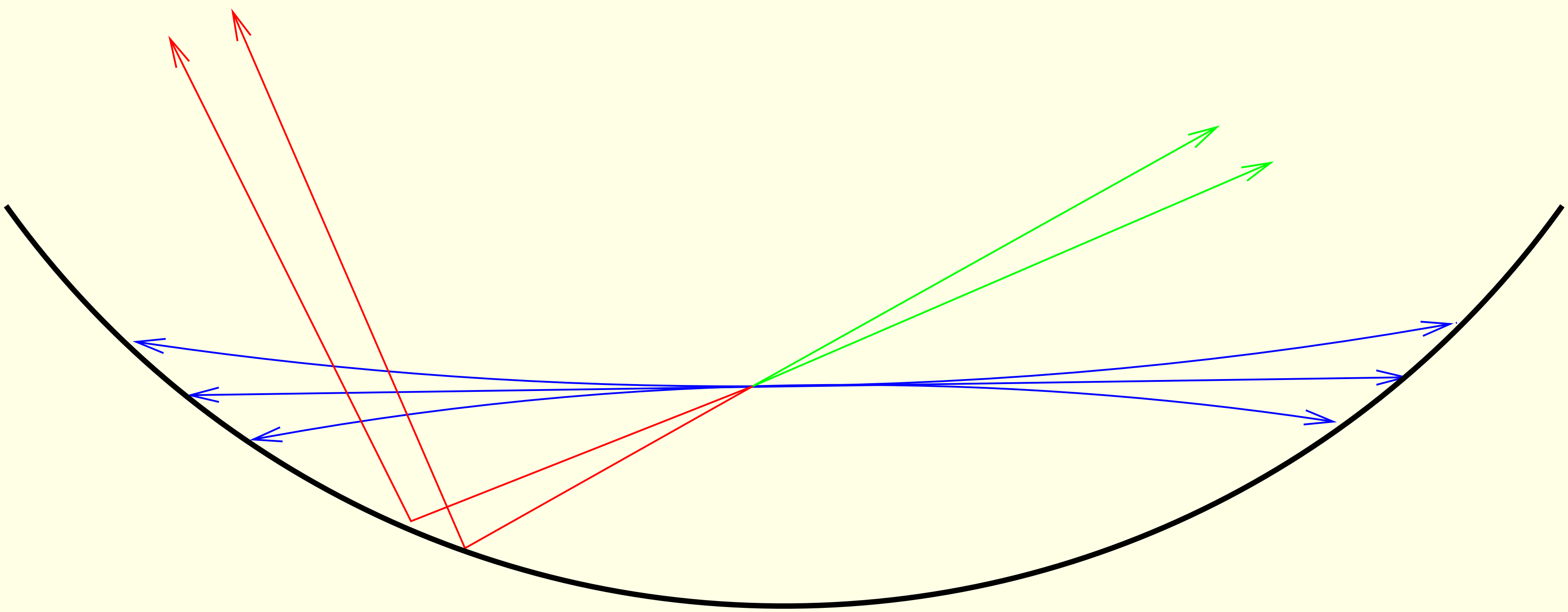
$$h^{1-d} \int_{\{x : \gamma(x) > h\}} \gamma(x)^{-1}\, dx \tag{21}$$

and one can prove that the contribution of the strip $\{x : \gamma(x) \leq h\}$ is $O(h^{1-d})$. Integral $\int \gamma(x)^{-1}\, dx$ is not converging but if the boundary is not very bad this is a logarithmic divergence.

To save the day one needs to apply old R.Seeley' idea: to extend time. Consider point (x, ξ) in the phase space. Then either trajectory launched from it is rather transversal to the boundary or almost parallel to it:



Outgoing (green), reflected (red) and tangent (blue) rays.

Even if rays diverge due to flow branching, the divergence of short rays is much smaller than their length.

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In the transversal case the trajectory launched in one time direction does not hit boundary for a while, and in the opposite time direction it hits boundary rather transversally and then again does not hit it for a while.

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In the transversal case the trajectory launched in one time direction does not hit boundary for a while, and in the opposite time direction it hits boundary rather transversally and then again does not hit it for a while. R.Seeley considered both cases but he was studied the asymptotics of $e_h(x,x,0)$ (actually he considered large eigenvalue asymptotics) but we are interested only in the asymptotics of $\int e_h(x,x,0)\,dx$ and it is easy to understand that it is sufficient to consider only one time direction for each pack of close trajectories and we take the direction from the boundary (green rays only).

In the parallel case trajectory does not hit the boundary for a while. So we trace everything for time $T(x)$ which is of the same magnitude as a length of a blue line on the picture (green rays keep away from boundary even longer).

Then the total contribution of the inner part $\{x : \gamma(x) \ge h\}$ does not exceed

$$h^{1-d} \int_{\{x:\gamma(x)>h\}} T(x)^{-1}\, dx. \tag{22}$$

R.Seeley considered C^∞-case when $T(x) = \epsilon\gamma(x)^{\frac{1}{2}}$; we assume that boundary is given by equation $\phi(x) = 0$ with $\nabla\phi$ disjoint from 0 and continuous with continuity modulus $|\log|x-y||^{-1-\delta}$ $(\delta > 0)$. Then $T(x) = \epsilon\gamma(x)|\log\gamma(x)|^{1+\delta}$. This makes integral $\int T(x)^{-1}\, dx$ converging and leads to remainder estimate $O(h^{1-d})$.

A bit more work - and one can recover remainder estimate $o(h^{1-d})$ under standard condition to Hamiltonian billiards (but one needs to consider reflections now!).

Analysis of systems is more tricky (because the geometry of propagation even away from the boundary is much more complicated) but also possible.

Details are published my paper

Sharp Spectral Asymptotics for Operators with Irregular Coefficients. II. Domains with boundaries and degenerations. Comm. Part. Diff. Equats., v. 28, no 1&2, pp. 125-156, 2003.

2 Microlocal Analysis

2.1 Propagation theorem

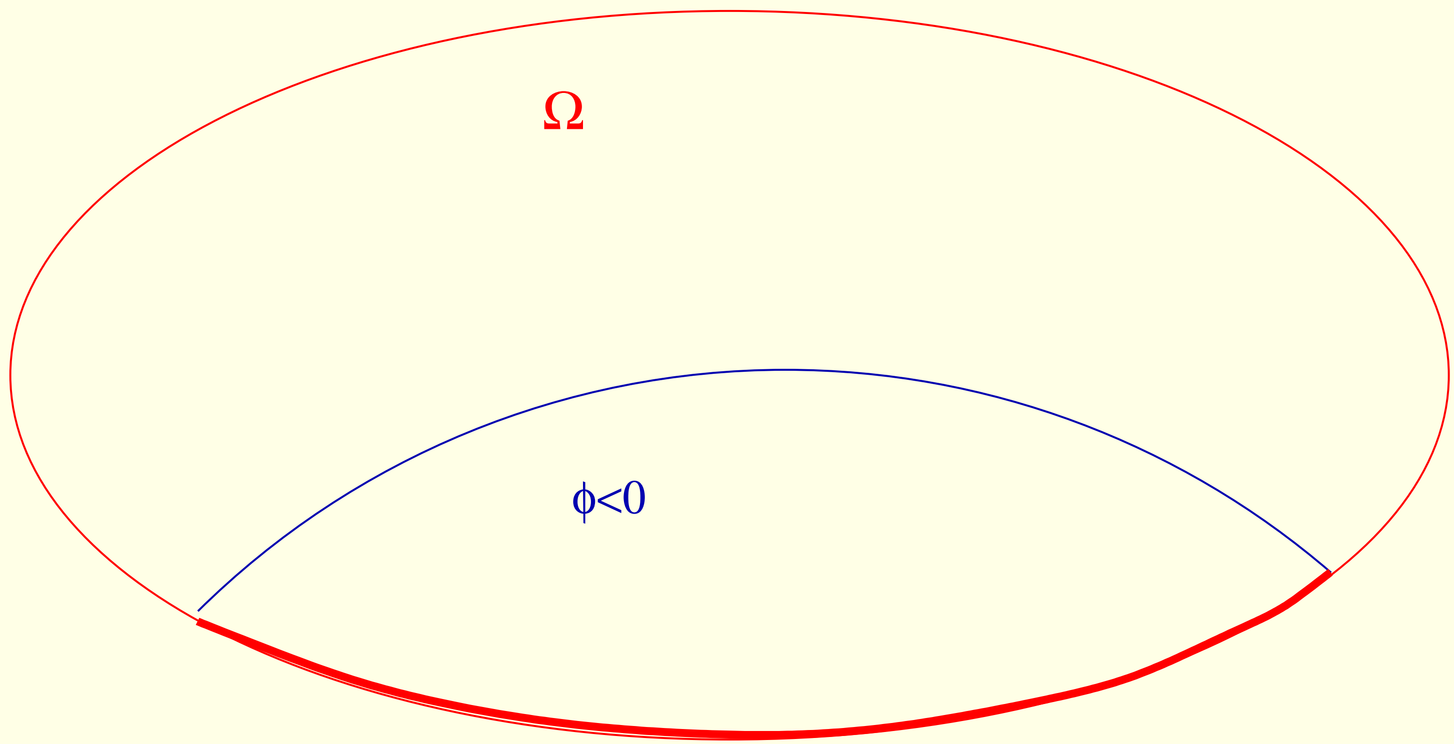
Propagation results proven by energy methods remind very much Holmgren uniqueness theorem and sounds like:

Theorem 3 *Let Ω be a bounded domain in the phase space, function ϕ satisfy microhyperbolicity condition and u be temperate.*

Further, let $Pu \equiv 0$ in $\Omega \bigcap \{\phi < 0\}$ and $u \equiv 0$ in $\partial\Omega \bigcap \{\phi < 0\}$.

Then $u \equiv 0$ in $\Omega \bigcap \{\phi < 0\}$.

Look at the picture:



Here u is temperate if $\|u\| \le Ch^{-m}$ for some m and u is negligible ($u \equiv 0$) in $\Omega^- = \partial\Omega \bigcap \{\phi < 0\}$ if $\|Qu\| \le Ch^s$ with some large s and any pdo Q with the symbol supported in Ω^-; similarly, u is negligible in $(\partial\Omega)^-$ if $\|Qu\| \le Ch^s$ for some pdo Q with the symbol equal 1 in the vicinity of $(\partial\Omega)^-$.

For scalar operators microhyperbolicity means exactly that $\frac{d}{dt}\phi \geq \epsilon > 0$ along Hamilton trajectories of P with $p(x,\xi) = 0$. For matrix operators definition is more complicated. Such theorem I first proved about 25 y.a. A bit later S.Wakabayashi found a geometrical interpretation in terms of generalized bicharacteristics. Similar theorems hold for boundary value problems as well.

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Proof starts from equality

$$2\text{Re }i(Pu,Qu) = \text{Re }i([Q,P]u,u) \tag{23}$$

if $P^* = P, Q^* = Q$. As Q we take Weyl quantization of $\chi(\phi(x,\xi)+\eta)\cdot\zeta(x,\xi)^2$ where $\chi(z) = 0$ for $z \geq 0$ and $\chi'(z) < 0$ for $z < 0$, $\zeta = 1$ in $\Omega\bigcap\{\phi < -\eta\}$ and supported in the small vicinity of it, $\eta > 0$ is a very small constant.

Then in the smooth case we get that $\|Q_1 u\| \le C h^{l+\delta'}$ where $\delta' > 0$ is some small exponent and Q_1 is a Weyl quantization of $\sqrt{-\chi'(\phi(x,\xi)+\eta)} \cdot \zeta(x,\xi)$ and we conjectured that $\|Q_2 u\| \le C h^l$ for some pdo Q_2 with the symbol equal 1 in the vicinity of $\{\phi(x,\xi) < -\eta\} \bigcap \text{supp } \zeta$.

Then in the smooth case we get that $\|Q_1 u\| \le Ch^{l+\delta'}$ where $\delta' > 0$ is some small exponent and Q_1 is a Weyl quantization of $\sqrt{-\chi'(\phi(x,\xi)+\eta)} \cdot \zeta(x,\xi)$ and we conjectured that $\|Q_2 u\| \le Ch^l$ for some pdo Q_2 with the symbol equal 1 in the vicinity of $\{\phi(x,\xi) < -\eta\} \bigcap \text{supp}\ \zeta$.

Then step-by-step increasing l by δ' and shrinking supports on each step, we prove that $\|Q_1 u\| \le Ch^s$.

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In application to spectral asymptotics $P = hD_t - A$.

The same arguments work in the case when all symbols are uniformly smooth in the scale ρ with respect to ξ and γ with respect to x, provided that standard microlocal uncertainty principle holds:

$$\rho \times \gamma \geq h^{1-\delta} \tag{24}$$

with arbitrarily small exponent $\delta > 0$.

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with arbitrarily small exponent $\delta > 0$.

Then we can pick up approximation parameter $\varepsilon = h^{1-\delta}$ and to prove sharp spectral asymptotics assuming that first derivatives of coefficients are Hölder continuous.

This is exactly what I did in the Spring 1999. Then I started to consider boundary value problems and think about further development but there was a question

Can we weaken (24)?

Let us look at PDOs which are "nice" in scale (ρ,γ) with respect to (x,ξ). This means that these pDOs are of the form $a(\frac{x}{\gamma},\frac{h\xi}{\rho})$ with smooth symbols a. Then the product of two such pDOs has a symbol

$$\sim \sum_{\alpha,\beta} \frac{1}{\alpha!\beta!} a^{(\alpha)}_{(\beta)} \cdot b^{(\beta)}_{(\alpha)} \cdot i^{|\beta|-|\alpha|} (\frac{h_{\mathsf{eff}}}{2})^{|\alpha|+|\beta|} \tag{25}$$

where $a^{(\alpha)}_{(\beta)} = (\partial_\xi)^\alpha (\partial_x)^\beta a$ and $h_{\mathsf{eff}} = \frac{h}{\rho\gamma}$.

Then (24) means that

$$h_{\mathsf{eff}} = \frac{h}{\rho\gamma} \le h^\delta \tag{26}$$

and restricting in (25) to $|\alpha|+|\beta| \le M_s$ we make negligible error.

So the question is

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We can do it assuming that derivatives of symbols are not growing fast with $|\alpha| + |\beta|$. Then we can take M depending on h. Since we need compactly supported functions, I tried Gevrey classes (we can pick axillary functions and mollifying function æ of any regularity). Proof of theorem 3 became tricky but I still managed; condition (26) became

$$h_{\text{eff}} = \frac{h}{\rho\gamma} \le |\log h|^{-1-\delta} \tag{27}$$

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$$h_{\text{eff}} = \frac{h}{\rho\gamma} \le |\log h|^{-1-\delta} \tag{27}$$

and I was able to pick up $\varepsilon = h\cdot|\log h|^{1+\delta}$ and to get sharp spectral asymptotics under assumption that first derivatives of coefficients are continuous with continuity modulus $O(|\log|x-y||^{-1-\delta})$.

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Can we do better than this?

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We need only derivatives of order $\le$ M

This changed the game because we can find functions supported in $[-1,1]$, equal 1 on $[-\frac{1}{2},\frac{1}{2}]$ and satisfying conditions

$$|a^{(\alpha)}| \le CM^{|\alpha|-m} + C \qquad \forall\alpha : |\alpha| \le M \tag{28}$$

These functions depend on M.

In this case restricting in (25) to $|\alpha| + |\beta| \le M$ we make an error $O((CMh_{\text{eff}})^M)$.

Taking optimal $M = \epsilon h_{\text{eff}}^{-1}$ we get an error $O(e^{-\epsilon' h_{\text{eff}}})$. To keep it smaller than h^s we need to assume that

$$h_{\text{eff}} \le C^{-1} |\log h|^{-1} \tag{29}$$

which is exactly equivalent to logarithmic uncertainty principle

$$\rho \times \gamma \ge Ch |\log h| \tag{30}$$

with C depending on s.

Proof of theorem 3 becomes really hairy but still possible.

Then we can pick up $\varepsilon = Ch|\log h|$ and to prove theorem 1.

C.Gerard noticed that this is very similar to L.Hörmander's approach to analytic wave front sets and it is true. Still, ideologically there is a big difference: L.Hörmander looked at ultra-fast decay with respect to h without rescaling while we want normal decay with respect to h (no use of anything better) but want the tightest scale possible. Also, L.Hörmander had not theorem 3.

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remained.

I began to harass my friends asking them some modified questions and learned that one of them, B.Paneah (Technion) recently proved (in different setting and it was much more general and precise result) that

If we can place a function in (ρ, γ) box in (ξ, x) (i.e. there exists v such that $\|v\| = 1$ and $v \equiv 0$ in $\{|x| \geq \gamma\}$ and $v \equiv 0$ in $\{|\xi| \geq \rho\}$ then ρ and γ must satisfy logarithmic uncertainty principle (30).

On the other hand, if (30) holds, function $e^{-sx^2\gamma^{-2}|\log h|}$ is "boxed" as $\rho\gamma \geq 2sh|\log h|$. So, I began to harass my friends asking them some modified questions and learned that one of them, B.Paneah (Technion) recently proved (in different setting and it was much more general and precise result) that

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On the other hand, if (30) holds, function $e^{-sx^2\gamma^{-2}|\log h|}$ is "boxed" as $\rho\gamma \geq 2sh|\log h|$. So,

Microlocal Analysis starts from the logarithmic uncertainty principle.

3 Future development

My Great Project © is to go through sharp results obtained for smooth operators in my book

Microlocal Analysis and Precise Spectral Asymptotics. Springer-Verlag, SMM 1998

and to prove them for not-so-smooth operators. I spent a year studying Schrödinger operator with the strong magnetic field in dimensions 2,3

$$A = \sum_{j,k} P_j g^{jk} P_k + V, \qquad P_j = hD_j - \mu V_j \tag{31}$$

($\mu \gg 1$). Here already difficult methods become ridiculously difficult.

Still I was able to prove that if g^{jk}, V and $F_{jk} = \partial_{x_j} V_k - \partial_{x_k} V_j$ have their second derivatives continuous with continuity modulus $|\log|x-y||^{-1}$ then smooth results remain valid.

This paper

Sharp Spectral Asymptotics for Operators with Irregular Coefficients. III. Schrödinger operator with a strong magnetic field.

can be downloaded from my web-site

http://www.math.toronto.edu/ivrii/Research/preprints

Now I am studying such operators in dimensions $d \geq 4$ and I am positive that I will be able to derive sharp spectral asymptotics which will be new even in the smooth case: new methods are useful for smooth case too.

Article

Sharp Spectral Asymptotics for Operators with Irregular Coefficients. IV. Multidimensional Schrödinger operator with a strong magnetic field. Full-Rank Case

can be downloaded from the same page.

Article

Sharp Spectral Asymptotics for Operators with Irregular Coefficients. V. Multidimensional Schrödinger operator with a strong magnetic field. Non-Full-Rank Case

is in progress.

4 Acknowledgements

1. Work was partially supported by NSERC grant OGP0138277.

2. Work was partially supported by Canada Council for the Arts via Killam Program.

3. This talk is actually BIRS-talk (March 2003) and MSRI-talk (April-2003) recycled.

5 Addendum: Questions

Q What happens if coefficients are even less smooth?

A Let us assume that $\vartheta(\varepsilon) = \varepsilon^l |\log \varepsilon|^{-\sigma}$ where either $l \in (0,1)$ or $l = 1$, $\sigma < 1$.

If $l = 1, \sigma > 0$ it means that the first derivatives of coefficients are continuous with continuity modulus $|\log|x-y||^{-\sigma}$; otherwise coefficients themselves are continuous with $|x-y|^l |\log|x-y||^{-\sigma}$. Then the approximation error does not exceed $C\vartheta(\varepsilon)$ and plugging $\varepsilon = Ch|\log h|$ we get $Ch^{l-d}|\log h|^{l-\sigma}$.

On the other hand, semiclassical remainder does not exceed

Ch^{1-d}/T with

$$T = \begin{cases} \epsilon & \text{for } l = 1, \sigma \geq 0, \\ \epsilon\varepsilon^{1-l}|\log\varepsilon|^{\sigma} & \text{otherwise.} \end{cases} \tag{32}$$

Really, in the former case $|\frac{d\xi}{dt}| \leq C$ and in the latter one $|\frac{d\xi}{dt}| \leq C\varepsilon^{l-1}|\log\varepsilon|^{-\sigma}$. This means that for time T given by (32), ξ and thus $\frac{dx}{dt}$ change their directions by no more than $C\epsilon$ and move away from the diagonal.

Still, $h^{1-d}/T \leq Ch^{l-d}|\log h|^{l-\sigma}$ and thus the final remainder estimate is $O(Ch^{l-d}|\log h|^{l-\sigma})$.

This lecture was prepared using TeX-power package for LaTeX

No animal suffered and no Micro$oft product was used in the process